VOLUME LIII

The Mathematics Teacher

NUMBER TWO

FEBRUARY 1960

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The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.*

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N. W., Washington 6, D. C.

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin, Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LIII, *number* 2 **February 1960**

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher*. Institutional subscription: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N. W., Washington 6, D. C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

*Mathematical education and the scientific revolution**

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An appreciation of mathematics is a cultural necessity and command of its techniques a practical occupational need.

EARLY IN THE nineteenth century, western civilization underwent changes of so drastic and rapid a nature that they have gone into history as the Industrial Revolution. Now, in the mid-twentieth century, we are living through a period which, if we emerge with flying colors, may well be christened the Scientific Revolution. Gradually, for generations, through theoretical advances so specialized and fundamental as hardly to impinge on the public intelligence, mankind has been preparing for an age of science. It is unlikely that the creative geniuses who paved the way could have foreseen, with all their imaginative powers, our present eruption into an age of nuclear energy, electronic brains, and sallies into outer space.

In its triple role as a tool, a language, and a mode of thought, the basic science of mathematics inevitably underlies and embodies the great work of Galileo, Newton, Einstein, and all the others who have led us, willy-nilly, to our contemporary level of scientific attainment. In our day, as never before, we have reached a stage where an appreciation of mathematics is a cultural necessity, and where a command of some of its techniques is, for an increasing proportion of the population, a practical occupational need. Culturally, the study of mathematics not only helps us appreciate a small part of our science-dominated world but also, if well-planned and executed, affords some insight into a magnificent and rapidly expanding creation of the human spirit. On the practical side, it is common knowledge that business, industry, governmental activities, the social sciences, and the natural sciences make growing demands on the mathematical equipment of technicians, engineers, and scientists. One phenomenon associated with contemporary society is the sharply dwindling market for unskilled labor and the consequently enhanced value of scientific and technical education.

Mathematics was and is derived by successive abstractions from the world of experience. Mathematical discovery is not guided by logic but by something which, for want of a better name, we call intuition. The role of logic is to verify, or sometimes to disprove, and to refine the conjectures based on intuition, but not to lead a mathematician to major new discoveries. A rigorous logical basis for the number system is less than a century old, but mathematicians had successfully worked with that system long before it had a satisfactory foundation. Such facts raise the question of the relative emphasis which should, in the secondary schools, be placed on (1) presenting a finished logical product to the students, and (2) guiding them along the fascinating paths leading from the world of things and events to the

* This article was stimulated by the author's participation in the Basic Curriculum Study of the Council for Basic Education.

abstractions of mathematics. Experience will help to resolve this question and to assess the merits of presenting certain levels of abstraction and degrees of logical rigor to students at various stages in their education.

New developments, both in mathematics and in the sciences dependent on mathematics, are slow to find their way into teaching. In our vast school system, there are inertial forces which dampen pedagogical progress into a tedious evolutionary development, far too slow to cope with the educational crisis accompanying the scientific revolution. Informed critics have often remarked that, with rare exceptions, we are teaching in our schools the mathematics of past centuries, paying no attention to modern developments, although few would contend that antiquity is a sufficient reason either to discard or to cherish any part of a curriculum. We propose, in this essay, to consider the study of mathematics without much regard to the relative novelty of different parts of the subject. We note, however, that some of the results of modern research are clearly relevant to the question of what should be taught in the schools. The critics quoted above are therefore justified in citing the exclusively antiquated subject matter in most of our classrooms as evidence of stagnation.

Among the modern phenomena which call for a re-examination of mathematical education are (1) new concepts and methods in pure mathematics, which can be used to promote an improved understanding and appreciation of the subject in the schools; (2) scientific and technological developments enhancing both the vocational and the cultural importance of the subject; (3) the widespread and expanding use of high-speed computers; (4) the steadily increasing utilization of mathematics in business, industry, and the social sciences, where the diversity of mathematical applications is a matter of astonishment to professional mathematicians and statisticians.

Even a well-planned and well-taught sequence of mathematics courses with exclusively traditional content may leave students with the belief that mathematics is a closed subject, with no important problems left to solve or, at least, with no prospect of solving them. The writer was not cured of that unfortunate impression until after he had begun a program of mathematical concentration in college. He was then amazed to learn how many thousands of pages of mathematics are published annually in this country alone.

Although the usual mathematical curriculum is old, it has undergone some alterations during the past two generations. The changes, of varying degrees of merit, have primarily involved new methods of presentation, rearrangement of topics, decreased emphasis on drill, new types of problems, and the omission of troublesome or seemingly unimportant material. During these developments, school mathematics has not been entirely at the mercy of textbook writers and educationists. The National Council of Teachers of Mathematics, for example, has consistently struggled and continues to struggle for the maintenance and strengthening of mathematical education. The same is true of a number of other organizations, committees, and individuals. Currently, there exist a few novel and decidedly promising organized efforts in which teams of classroom teachers and university mathematicians are working in close co-operation to produce new curricula, together with teaching materials, adapted to the educational needs of America in the mid-twentieth century. More will be said below concerning some of these efforts.

Many laymen might derive from the public press the impression that Soviet satellites suddenly roused us from complacent lethargy and galvanized us into frantic efforts to achieve a new scientific level. On the contrary, almost all of the current projects for improving mathematical education were well under way before the first Sputnik and were not modi-

fied as a result of it. The more recently-initiated efforts also show no signs of having been primarily motivated by Soviet achievements. But we still owe to the USSR a certain gratitude for unwittingly launching, along with its first satellite, a powerful American clamor for improved scientific education. The panic reaction to Sputnik was not surprising in a nation which, with some justification, has always regarded itself as unchallenged leader of the world in science and technology. It was, furthermore, made abundantly clear to the American public that the launching of the first artificial satellite was not an isolated tour de force. While it was accomplished by an older generation of Soviet scientists, the quality of their work is likely to be maintained as a result of the sound mathematical and scientific training offered by the Soviet educational system. The fact that Soviet education is oriented toward service to the state does not necessarily mean that it will be less effective than our own. Soviet energy and drive have brought the USSR to a point of offering keen competition in this critical area, with an expectation of even keener competition in the future.

We, with our cherished freedom to plan our lives, could not, if we would, imitate the dictatorial Communist educational system. Nor are we quite ready to be frightened by the Communists into the maximum effort consistent with democracy to divert our youth into scientific careers. Indeed, every scientific or technological group which the writer knows to have expressed itself on the subject has overwhelmingly opposed any attempt to warp our culture artificially in favor of science or engineering at the expense of the humanities. The time and the public mood are, however, ripe for reform, and it behooves scientists, scholars, and educators to take advantage of that circumstance.

The struggle of mathematicians for educational reform extends back as far as the author can remember, and much farther. In recent decades, until a few years ago, they fought the rear guard battle of a retreating army. Mathematics, like other academic subjects, could not escape the deleterious effects either of "life adjustment" theories or of mass education, however proud we, as a country, may be of the latter. In our rapid march toward the enviable state where it is natural for almost every American to complete high school, we were confronted with the overwhelming problem of catering to all levels of scholastic ability and inability, combined with every type of educational motivation or the lack of it. The problem was poorly resolved. At best, it would have been extremely difficult to satisfy the well-motivated and capable students in the face of the gigantic task of coping with the vast and sudden increase in the school population as a whole. The difficult became impossible in the dim, flickering light of some of the prevailing educational theories with nonacademic goals, and the superior students were often neglected in a program dedicated to mediocrity. As a result, much of the more challenging material was gradually eliminated from the schools, the amount of homework dwindled away, and the proportion of students, even able students, studying secondary school mathematics was severely reduced. We are thus faced with the question of what to restore of the material which fell by the wayside, in addition to deciding what new topics to introduce and what currently taught items to omit or de-emphasize.

In such a discussion as this, it is natural to stress deficiencies in need of correction and thus, perhaps, to give an impression more predominantly negative than even the sad facts warrant. To offset this tendency, in part at least, we remark that our country contains many schools—in our opinion an increasing number—with excellent mathematical programs. They represent exceptions to most of the adverse criticisms directed at our schools as a whole.

It is no easy matter to determine just what curricular changes are needed to supplement and modernize mathematical education. A few general trends clearly suggest themselves, although heated arguments take place on the merits of including or omitting certain topics. Such differences of opinion stem largely from differences both in mathematical interests and in educational outlook. It is neither possible nor desirable to terminate these controversies. Our educational system can benefit by vigorous debate and by diverse experimental studies tending to resolve some of the points at issue. A great deal of hard work, bold experimentation, and thoughtful study will indeed be needed to clarify the picture of what we should and can do by way of curricular reform. Fortunately, several groups have embarked on the task, with highly competent, enthusiastic personnel and with apparently adequate financial resources. Even more fortunately, we need not mark time while awaiting the culmination of their efforts, for it is more urgent to strengthen our program, even along traditional lines, than to introduce drastic changes. There is a large measure of agreement on the topics truly essential to high school mathematics. Those on which experts sharply disagree can safely be regarded as optional.

One reason why rapid reform cannot occur along a broad front is the fact that, as everyone concerned is keenly aware, all significant innovations in school mathematics lead to serious problems in the education and re-education of teachers. Appropriate training for them is the most conspicuous and difficult prerequisite for important reforms in secondary education. Curricular innovations demand that they acquire a new point of view and new knowledge. Something along these lines can be brought about without interruption of work by the in-service training of experienced teachers; even more can be accomplished by their attendance at summer institutes; and still more by their participation in academic-year institutes imbued with the spirit of contemporary mathematics. As proposed curricular changes win sufficient acceptance, they are sure to be included in the programs of some of the institutes sponsored by the National Science Foundation. The conspicuous, almost overwhelming, interest of teachers and schools in the reform movements and the institutes is a most encouraging augury. Such enthusiasm has greeted the latter that the demand by worthy candidates for admission far exceeds the number the institutes can accommodate. This does not necessarily mean that there should be more or larger institutes. In considering the merits of expanding, one would need to take into account (1) the problem of staffing the institutes, and (2) where academic-year programs are at stake, the question of how many mathematics teachers per year should be temporarily withdrawn from their classrooms. As matters now stand, a small but significant proportion of the mathematics teachers from all sections of the country attend institutes and return to their work with new knowledge of subject matter and methods. Subject matter is by far the principal concern of the NSF institute program.

In what follows, we have secondary school mathematics largely in mind, even though curricular problems at more elementary and more advanced levels are also acute. It is at the secondary level that the most interesting activities are well under way. Problems in elementary mathematical instruction present special difficulties, and efforts to deal with them are in a comparatively primitive stage. However, they are being experimentally studied by competent persons and will soon be systematically attacked on a larger scale. Between the elementary and the high school level, we note that the University of Maryland is active in a promising program to strengthen mathematical instruction in grades seven and eight. Beyond the secondary level, the Committee on the Undergraduate Program, established by

the Mathematical Association of America, has been occupied for several years with the question of modernizing the college curriculum in mathematics.

As a generally acceptable working definition, we can say that, in this country, college mathematics commences with analytic geometry and the calculus, either taught as separate courses or in combination with one another. All the preliminaries belong in the secondary schools, however prevalent and temporarily necessary may be the practice of offering some of them in college to correct earlier deficiencies. As an immediate project, let us take the strongest steps we can toward the goal of ensuring that every qualified student in America shall have the opportunity, before completing high school, to master at least the traditional algebraic and geometric topics and to prepare, if college-bound, for the study of analytic geometry and calculus. This is a specific major objective on which we should fix our sights and toward which we can continually strive until it is reached. It is an attainable goal, although the path to it appears long and difficult. Our American system of decentralized control of the schools makes the attainment of such an objective a matter for local action in each community where the mathematical program is inadequate. Local action, in turn, will take place only after a community has become aware of and aroused to its need. Even if we should not everywhere succeed in attaining this first important goal, we shall have the satisfaction of knowing that a precious educational gain accrues with every community where it is achieved.

While the foregoing paragraph emphasizes preparation for college mathematics, it should be remarked that a serious student, even if not college-bound, will derive more benefit from good courses including the necessary background for calculus than from any of the terminal secondary school courses known to the writer. The latter, however, have their place, especially for students lacking in the ability to benefit by more substantial courses. Unfortunately, many of our students who possess such ability feel no incentive to take advantage of the more valuable and demanding studies. Everything possible must be done to motivate them. Regardless of their vocational interests, or even if they have developed none, all capable students should be encouraged to go further in mathematics than the traditional year each of algebra and geometry. Teachers, counselors and parents have a direct responsibility to see that students do not handicap themselves through ignorance of the cultural and practical importance of mathematics in the modern world. Four years of high school mathematics are generally to be strongly recommended for capable students. In this area, as throughout the curriculum, there is great need for intelligent guidance. It is a crying shame, both for the individual victims and for our society, that so high a proportion of the more capable students are allowed to drift peacefully through the public school system and out into the world without being stimulated either to prepare for higher education or to benefit by the more valuable courses available to them in the secondary schools. While we Americans believe in freedom of choice, including the freedom to make bad decisions, we should also recognize the duty of the school system to see that students have the opportunity to select their courses from an excellent educational menu, guided by the best possible advice and information.

During the remainder of this discussion, we concern ourselves only with the needs of serious, well-motivated students of average ability or better. The others have long received at least their full share of special attention from educators, even though the courses offered to them are likewise in critical need of revision.

It is possible, without undue effort, to list the topics essential to high school mathematics, regardless of whether they are to be grouped along traditional lines into separate courses in geometry, algebra,

and trigonometry, or whether they are to be combined into unified courses. A good instructional job can be done in either way. Furthermore, the essential topics are not so demanding as to require for mathematics an exorbitant share of a well-balanced high school program, in which all the various academic subjects, extracurricular activities, and other services receive due attention. Indeed, many high schools even now, without overemphasizing mathematics, not only cover the essentials but include a number of optional topics in their regular courses, and, in addition, offer mathematical electives for those with special interests. It is not part of the present purpose to give detailed lists of topics for secondary school instruction. Several good lists are in existence, and others will be produced for the guidance of teachers and administrators. They will vary with changing conditions and will differ according to the predilections of those who compile them, but lists emanating from competent sources will be in general agreement on truly essential topics. There is no unique method of getting adequate preparation in mathematics, any more than there is a unique good auto route from New York City to Los Angeles, although all appropriate routes have something in common.

In this time of political, scientific, and educational turmoil, much is being said and written in favor of radically new approaches to the study of mathematics and in favor of introducing novel mathematical material into the program. This fact, together with a public demand for strengthening scientific education, might induce an administrator to undertake drastic and premature revisions in curriculum or to embark on experiments for which his staff is ill-equipped. We do not recommend that every high school plunge into curricular experimentation in mathematics. Few teachers have the desire or the mathematical background to do so with either pleasure or profit. It is far better for all concerned that mathematics teachers, who are generally among the strongest advocates of high standards, be given the opportunity to do well what they have been trained to do rather than be expected to do something strange and spectacular. Most schools should strive for the development of mathematical insight and understanding through existing courses, while awaiting the necessary conditions for successfully introducing novel features into the curriculum. In particular, it is desirable that major innovations await the publication of some of the results of the principal experiments and studies now being carefully conducted by teams of classroom teachers and university mathematicians. These projects will produce valuable information concerning topics which prove to be successful in the classroom. They will also provide instructional material for classroom use and for the information of teachers with regard to new subject matter and methods of presenting it.

As a preliminary to discussing some of the reforms currently under consideration, we list the principal groups whose primary objective is the improvement of secondary school mathematics.

1. The Commission on Mathematics. This Commission was established about four years ago by the College Entrance Examination Board. A report presented by its chairman, Professor A. W. Tucker of Princeton University, to the 1958 International Congress of Mathematicians included the following statement of purpose. "The purpose of the Commission on Mathematics is to bring about the introduction into as many American secondary schools as possible within the next few years of a revised mathematics college preparatory curriculum oriented to the needs of the second half of the twentieth century." The Commission has clearly formulated the broad outlines of its recommendations for a four-year program of high school mathematics, without elaborating them into detailed curricula. The published *Report of the Commission* con-

tains a careful statement of these recommendations and of the reasons for making them. A substantial volume of appendices is especially designed for the in-service training of teachers wishing to become familiar with the relevant mathematical concepts.

2. School Mathematics Study Group. This is a national project with headquarters at Yale University and with the support of the National Science Foundation. It was established in the spring of 1958 and held its first session early that summer. Teams of classroom teachers and university mathematicians are working in close cooperation to produce sample curricula, classroom materials, and expository tracts for the improvement of secondary school mathematical education, and also of the mathematics programs at more elementary levels. The SMSG has very carefully studied the recommendations of other major groups with related objectives. The proposals of such groups fortunately agree in their general direction, though, of course, not in all details. The SMSG will produce material designed to help teachers improve themselves, so that they can, in their classrooms, implement various recommendations for instructional reform. It also has a monograph project, whose major purpose is to enable students to obtain a better view of the scope of mathematics and of the fact that mathematics is alive and growing.

3. The University of Illinois Committee on School Mathematics. This project began in 1951 and, since July 1956, has received substantial support from the Carnegie Corporation. It has developed student texts and teacher guides which were used in 1957–58 in twelve schools by forty teachers and about seventeen hundred students. In 1958–59, the program included more than fifty schools, more than one hundred teachers, and more than four thousand students. The project can be described as advanced curricular research. Its program has led to bold and interesting experimentation, including the use of fairly advanced abstract mathematical material at the high school level. The UICSM texts cover four full courses and are designed for students of better than average ability.

One of the general difficulties from which American students suffer is that so little is expected of them. In mathematics, more homework should be demanded, and it should be made impossible for anyone, however great his native ability, to loaf his way through a mathematics course, though the subject will always be easier for some than for others. The need for more homework, however, is not to be construed as a call for introducing or reinstating the practice of requiring large doses of routine computational problems. On the contrary, we should recognize that the development of high-speed electronic computers has made obsolete the stress which used to be laid, for example, on computing the roots of numbers, figuring successive approximations to solutions of algebraic equations, doing routine problems in logarithmic computation, and solving numerous triangles to several significant figures. Probably the most widely accepted proposed revisions of school mathematics are (1) the elimination, or at least the drastic reduction, of drill in such computational problems, and (2) an increased emphasis in trigonometry on those interesting properties of functions of the general angle which lay the foundation for important uses of these functions in both pure and applied mathematics at a more advanced level. The Commission on Mathematics recommends the introduction in grade eleven of fundamental trigonometry, centered on coordinates, vectors, and complex numbers. Even the student who does not continue with mathematics will derive more benefit from such studies than from solving triangles. There is no need to sacrifice thereby the advantages of computational work, for it is a simple matter to accompany the new material with problems which challenge the student and develop his technical skill.

Two mathematics courses whose restoration or retention has been challenged are (1) solid geometry and (2) the once-traditional treatment of Euclidean plane geometry. The latter, to be sure, has the merit of being a good example of a mathematical discipline in the form of a structure of undefined terms, axioms, theorems (deduced more or less rigorously from the axioms), and concepts defined in terms of them. It has traditionally been the only such example regularly presented in the secondary schools. Its disadvantages include the devotion of an unnecessary amount of time and effort to just one such structure. It then leaves the student with the impression that, while Euclidean geometry is a rigorous and to some a beautiful logical edifice, the same is not and never can be true of algebra and arithmetic. By presenting parts of Euclidean plane geometry and, in addition, other samples of the axiomatic method, selected from both algebra and geometry, it is possible to enhance the benefits of the traditional geometry course while eliminating its disadvantages, and to do so in a more interesting, even entertaining, fashion. Euclidean solid geometry, where taught, has continued the work of plane geometry. It appeals strongly to those few who have a genuine flair for the subject. A number of engineering schools have required it for admission, and some still do so, largely because of the assumption that it develops a three-dimensional perception needed in certain basic engineering studies. That perception can, however, be acquired in other ways. The Commission, for example, would include some solid geometry with plane geometry in a revised tenth-grade geometry course, employing both synthetic and analytic methods according to convenience. The position of plane geometry is doubtful; it is even more doubtful that an entire course should be devoted to solid geometry. There is a current tendency to drop it as an engineering requirement.

With regard to high school algebra, there are influential proposals that inequalities and deductive reasoning be included, so as to emphasize structure rather than manipulative skill.

Among the more controversial innovations has been the introduction, at various levels, of material derived from modern set theory and from mathematical logic, including an approximation to a logical development of the real number system. Certain portions of set theory, suitably presented, are close to the world of experience. They can be appreciated by quite young children, and they are of real assistance in developing the number concept. Many other pure mathematical abstractions likewise seem to be understood and to arouse considerable interest among high school students, at least when presented by an outstanding and enthusiastic teacher. The depth of appreciation is, however, hard to assess, and it may be that only a small minority of students really profit by some of the more ambitious abstractions until they have acquired a mathematical maturity which one cannot legitimately hope to find at the secondary school level.

Whatever objections may be raised to some of the more radical innovations currently being proposed, they at least help to make mathematics appear in its true light as a living, growing creation of the human spirit, worthy of study for its own sake and not merely in its subordinate role as a tool for the use of grocers, actuaries, engineers, and physicists.

Where the staff and student body are adequate, elective mathematics courses are valuable for students with special interests. For example, some schools have courses in analytic geometry and calculus at the usual beginning college level, and some colleges accept such courses either for credit or for advanced standing.

The Commission on Mathematics has suggested as electives for the last year of high school a semester course in elementary analysis to be followed by a semester course in probability with statistical applications. Some schools are offering the

latter, using an experimental text, *Introductory Probability and Statistical Inference for Secondary Schools,* which was prepared under the auspices of the Commission.

Students with genuine mathematical interests should be encouraged to activity in school mathematics clubs and in various competitions—for example, the competitive examinations sponsored on a statewide and national (even international) basis by the Mathematical Association of America. School teams and individuals can take part. It is to be hoped that public recognition as well as material rewards (in the case of scholarship competitions) will increasingly endow intellectual achievements with some of the same kind of prestige which has long been accorded to outstanding athletic performance.

Through the combined influence of the institutes, the groups mentioned above, and others, we can hope for a rising level of mathematical instruction in the schools and, eventually, a well-considered modernization of the curriculum. A strong fundamental high school program, supplemented by interesting electives and extracurricular mathematical activities, will stimulate the imaginations of receptive students, from whose ranks must come the scholars and scientists of tomorrow.

What's new?

BOOKS

College

Logic in Elementary Mathematics, Robert M. Exner and Myron F. Rosskopf. New York: McGraw-Hill Book Company, 1959. Cloth, xi+274 pp., $6.75.

Real Analysis, Edward James McShane and Freeman Arthur Botts. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1959. Cloth, ix+272 pp., $6.60.

Selections From Modern Abstract Algebra, Richard V. Andree. New York: Henry Holt and Company, 1958. Cloth, xii+212 pp., $6.50.

University Mathematics (2nd ed.), Joseph Blakey. New York: Philosophical Library, 1959. Cloth, 581 pp., $10.

Secondary

Basic Geometry (3rd ed.), George David Birkhoff and Ralph Beatley. New York: Chelsea Publishing Company, 1959. Cloth, 294 pp., $3.95.

Notes and Exercises on Coordinate Geometry, I. Tomlinson. London: Methuen and Company Ltd., 1959. Cloth, 128 pp.

Miscellaneous

Liberal Education in the Professions, Earl J. McGrath. New York: Bureau of Publications, Teachers College, Columbia University, 1959. Paper, vi+63 pp., $1.50.

Mathematics and the Physical World, Morris Kline. New York: Thomas Y. Crowell Company, 1959. Cloth, ix+482 pp., $6.

The Scientific American Book of Mathematical Puzzles and Diversions, Martin Gardner. New York: Simon and Schuster Publishers, 1959. Cloth, xi+178 pp., $3.50.

Studies in Mathematics Education: A Brief Survey of Improvement Programs for School Mathematics. Chicago: Scott, Foresman and Company, 1959. Paper, 57 pp., 50¢.

BOOKLETS

Secondary

Essentials of Solid Geometry, A. M. Welchons, W. R. Krickenberger, and Helen R. Pearson. Boston: Ginn and Company, 1959. Paper, 124 pp., $1.20.

Graphs: Columnar, Straight Lines, Locus, Curved and Complex (3rd ed.), R. J. Gillings. Sydney: Australasian Publishing Co., 1959. Paper, 77 pp., 7/6*d*.

Mathematical Tables: Designed for the Use of Secondary Schools, High Schools, College Students and University Undergraduates (rev.), R. J. Gillings and V. R. Outten. Sydney: Australasian Publishing Company, 1959. Paper, 33 pp., 2/3*d*.

The Ball State experimental program

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Seventh- and eighth-grade mathematics—not social applications, classical algebra taught with precision and meaning, and Euclid made precise and rigorous according to the standard of modern mathematics.

INTRODUCTION

THE 1959–60 SCHOOL year is the fifth year of the Ball State experimental program in plane geometry, the third in beginning algebra, the second in eighth-grade mathematics, and the first in intermediate algebra. Text materials in these areas have been prepared by Professors Merrill Shanks of Purdue University and Charles Brumfiel and Robert Eicholz of Ball State Teachers College.

The geometry text is completed. The beginning algebra is almost in final form. Text material for the eighth-grade course has taken definite shape but will undoubtedly be subjected to some revision. The development of the intermediate algebra has only begun.

The Ball State laboratory school, Burris, has served as the chief testing ground. In addition several Indiana high schools have co-operated during the past three years by using the geometry, algebra, and eighth-grade materials. Approximately fifteen teachers are presently involved in this experimental program.

The experimental classes have for the most part been constituted of students of average ability. The teachers who volunteer for experimental teaching assignments have, in general, no particular training for the task; but during the school year they meet periodically for a discussion of pedagogical problems. In addition Professor Eicholz regularly visits the classrooms of some of the participating teachers.

A significant feature of both the algebra and the geometry is a full unit on logic in each. In these chapters the student is taught to understand some of the basic forms of logical reasoning without trying to develop an elaborate symbolism. Thus, the only symbol introduced is the arrow "→" for implication. However, in order to have language to talk about proof it *is* necessary to define, and use, some logical terms; for example, conjunction, disjunction, negation, implication. These terms are applied both to everyday situations and to simple problems in algebra or arithmetic in order to clarify the concepts. In general the students enjoy this work. The "trick" in teaching is to let each student go as deeply as his abilities permit. Unlike many topics in algebra, this is not drill material.

It should be made clear that these units on logic are full chapters. They take at least two or three weeks in both courses when done thoroughly. The reason for this is that once the topic is begun there is no easy place to quit without leaving things in mid-air.

Every effort is made in both algebra and geometry to develop the student's power to make proofs from clearly-stated assumptions. The development of such power, with accompanying mathematical imagination, is of course a slow and continuing process. We think that time spent

on the logic "pays for itself" in the course of the year. Of course it is expected that the teacher will call attention to points of logic as they occur in geometric proofs.

An evaluation of the Ball State program at present would be premature. Not until the fall of 1961 will we send to college a group of students who have been exposed to more of our program than the single year of geometry.

We who are closest to the program can make a few tentative observations.

1. Teachers who have a strong background for teaching the experimental program are enthusiastic about it.
2. Teachers with weak backgrounds have mixed reactions. They experience some doubts and frustrations. Most of these teachers, however, feel that as they develop the particular skills and understandings required to teach the experimental material they will be able to teach it more effectively than the traditional material of the past.
3. Capable students who receive moderately competent instruction are usually enthusiastic about the course. It has been particularly gratifying to see several brilliant students, who had found conventional mathematics dull and uninteresting, come to life after exposure to the new materials and perform magnificently.
4. Weak students who cannot follow patterns of precise logical reasoning seem to perform as well in the experimental courses as in traditional courses. Their attitudes toward the experimental courses seem little different from the attitudes of weak students toward conventional mathematics courses.
5. Within the group of average students there is considerable variation. Many of these students rise to the challenge and do beautiful work. Others passively resist attempts to arouse their interest, refuse to work hard, and perform poorly. Clearly, the central ideas of the geometry and algebra are not inaccessible to the high school student of average ability who is reasonably well-motivated and has good work habits.
6. During these past five years our philosophy of textbook writing has jelled. We consider it necessary to treat students as if they are, intellectually, adults. Whenever we have tried to gloss over a subtle point our teaching experience has compelled us to rewrite and to be completely candid. We delete archaic language entirely and use modern language consistently.

The eighth-grade course

The primary objective here is to lay a foundation for the continuing study of mathematics. This is *not* a course in how to write checks—how to read gas meters—how to make and interpret circle and bar graphs. It is *not* written as a terminal course that presents such topics as insurance, taxation, installment buying, and the mathematics of cookery and sewing for those recalcitrant inhabitants of our schools who are only waiting for time to release them from their school custodians.

Roughly speaking, topics covered in the eighth grade-course may be classified under three major headings: Systematic Arithmetic with Applications, Mathematics for Fun, and Introductory Algebra.

In the first units the role of symbolism in mathematics is considered. The distinction between a symbol and the *concept* that the symbol represents is emphasized. The use of parentheses to indicate the order of operations is explained. Such relations as equality, inequality, greater than, and less than, with the accompanying symbols, are introduced and utilized in problem sets. A small study of the history of number symbols is made, and careful attention is given to the concept of place value within both our decimal system and systems of numeration employing other bases.

A systematic study is made of the arithmetic of the set of numbers (0, 1, 2, 3, 4, $\cdots$). In introducing the properties of addition and multiplication in this set, emphasis is placed upon student dis-

covery. The commutative, associative, and distributive laws are formulated as general principles. They are not presented as postulates.

Attention is called to the subsets designated by "even," "odd," and "prime," and a little number theory is studied. Students are encouraged to make informal proofs by setting before them problems like:

> The product of two numbers is odd. What can you say about these numbers?
>
> Explain why, if you square any number and add this product to the number itself, you will have an even number.
>
> The difference between the primes 5 and 2 is 3. Explain why no other pair of primes has this property.
>
> What can you say about the set of remainders obtained when one divides each of the primes greater than 3 by 6?

The introduction of the set of rational numbers is preceded by a short unit that would be classified as problems in ratio and proportion. This unit belongs in the "for fun" category. None of the terminology of ratio and proportion is employed. Students are shown by examples how to describe various situations by *number pairs*. For example, the symbol (6, 1) may be used to indicate that in a certain orchard there are six apple trees for each plum tree. The symbol (3, 2) compares the number of peach trees to plum trees. Now we ask for a symbol that compares the number of apple trees to peach trees.

Problems of the following type occur.

> The number pair (3, 4) compares the sizes of two numbers.
>
> (a) If the second number is 20, what is the first?
> (b) If the sum of the numbers is 21, what is the first?
> (c) If the difference of the numbers is ten, what is each number?
> (d) If the product of the numbers is 192, what is each number?

Our experience in teaching this unit indicates that these problems stimulate a great deal of interest and lead students to invent for themselves techniques which are equivalent to the rules ordinarily given for the manipulation of ratios and proportions.

In the work with rational numbers it is observed that the basic principles formulated for the whole numbers hold also for this new set.

Irrational numbers are presented as infinite decimals. An informal proof that the square of no rational number is two is presented. The identification of the set of rational numbers with the set of repeating decimals is made, and computational techniques for expressing repeating decimals as fractions are developed.

The work in algebra utilizes those set theoretic concepts that are serviceable in the discussion of the solution of equations and inequalities. Variables are first used to construct open sentences referring to sets containing only a few elements. For example, after agreeing that all variables refer to the set (0, 1, 2, 3, 4), the student is asked to determine the solution set for sentences like

$$r>2; \qquad n<8; \qquad t+4 \text{ is in the set};$$
$$a+a=2\cdot a; \qquad b+5=3.$$

Sentences true for every number in the set of reference are called generalizations, and some algebraic activity is motivated by a search for generalizations.

Sentences in two variables are considered, and their solution sets are seen to be subsets of a set of ordered pairs. Again in these preliminary stages the variables are restricted to finite sets. The student is introduced at this point to the graphing of systems of equations. Gradually the algebra takes on an "orthodox" appearance. Negative numbers are introduced and equations and inequalities are solved over the full set of integers and then over the signed rational numbers. The applications and word problems are much like those in a conventional text.

THE ALGEBRA COURSE

The algebra text represents an attempt to teach classical algebra with precision and meaning. Postulates, definitions, and theorems are presented, but the formal structure remains in the background. This

logical structure is taught less for its own sake than for the purpose of illuminating the many calculation rules of arithmetic and algebra. Our goal at the end of the ninth-grade work in algebra is a student who has firm techniques and the ability to deduce most of these techniques, *using numerical examples*, from the postulates and definitions. Thus a proof, based upon the distributive law, that $(-2)(-2)=4$ should be possible for many students.

The algebra is presented as an extension or generalization of arithmetic with all its familiar practical applications. This does not mean superficial manipulative ability; it means skill with understanding. As a consequence of this view much time is spent illustrating the power of the basic laws. These laws are formulated as postulates. Since a systematic study of logic has given students some insight into the nature of proof, it is now clear to them that the starting point for any logical development must be a body of assumptions. The postulates are stated first for the set of non-negative integers.

$$\{0, 1, 2, 3, \cdots.\}$$

1. For each ordered pair a, b, $a+b$ and $a \cdot b$ exist and are unique.
2. For all a, b, $a+b=b+a$ and $a \cdot b=b \cdot a$.
3. For all a, b, c, $(a+b)+c=a+(b+c)$ and $(a \cdot b) \cdot c=a \cdot (b \cdot c)$.
4. For all a, b, c, if $a+b=a+c$, then $b=c$; if $a \cdot b=a \cdot c$ and $a \neq 0$, then $b=c$.
5. For all a, $a+0=a$, $a \cdot 1=a$, and $a \cdot 0=0$.
6. For all a, b, c, $a(b+c)=a \cdot b+a \cdot c$.

Students see little reason for making a great fuss over the above assumptions for the set of non-negative integers. However, the solution of equations is made to rest upon these postulates and some appreciation for their usefulness begins to develop. Considerable interest is aroused, also, when we point out that the entire year's work will be based upon this one set of postulates. *No additional memorizing is necessary.*

A special study of the subtraction operation is made prior to the introduction of negative numbers. Subtraction is defined in terms of addition ($a-b=c$ *means that* $a=b+c$). The fact that subtraction is not always possible motivates the extension to the integers.

The student recognizes the power of the basic postulates for the first time when he sees them used to establish by proof the rules for computing with the integers. We postulate that for every natural number a there exists a new number, $-a$, called the negative of a, and such that

$$a+(-a)=0.$$

We call the set, consisting of zero, the natural numbers and the negatives of the natural numbers, the *integers*, and we extend the scope of our assumptions by postulating that our old laws hold for this set.

Now, familiar rules for calculating with integers fall out as theorems. For example, if a and b are non-negative integers, then

1) $a+(-b)=a-b$ or $-(b-a)$ according as $a \geq b$ or $a<b$.
2) $(-a)+(-b)=-(a+b)$.
3) $a(-b)=-(ab)$, and $(-a)(-b)=ab$.
4) $(-1) \cdot a=-a=0-a$.

After the experience of seeing the rules for calculating with integers established firmly by proofs, students never question the need for assumptions. While they worked with the non-negative integers, in the background undoubtedly was the feeling that the postulates were not really necessary. They *knew* how to compute with these numbers. But now the postulates enable them to perform new calculations. They recognize that deductive reasoning, based upon a set of assumptions, has extended their mathematical knowledge and led to the discovery of new facts.

The next extension of the integers to the rationals finds the students quite co-operative. Having postulated that for every pair of integers a, b with $b \neq 0$ there exists a rational number a/b such that

$$b \cdot \frac{a}{b}=a$$

and having also extended the scope of our

basic laws to govern the algebra of this new set, students present correctly proofs of such theorems as

$$\frac{1}{2}+\frac{1}{2}=1; \qquad 6\cdot\frac{1}{2}=3;$$

$$\frac{1}{2}=\frac{2}{4}; \qquad \frac{a}{b}+\frac{c}{b}=\frac{a+c}{b};$$

$$\frac{a}{b}=\frac{c}{d} \text{ if and only if } ad=bc.$$

Even the best of these students have thought of the arithmetic of fractions as based upon meaningless rules. There is almost universal appreciation of the fact that these rules are inevitable consequences of our basic assumptions.

Having available the algebra of the rational number system, many conventional topics are now considered—calculation with rational expressions, systems of linear equations, factoring of polynomials over the rational field, simple quadratic equations, ratio and proportion, and various applications.

A chapter upon sentences, relations, graphs, and functions stresses heavily logical and set theoretic concepts. This chapter has as its goal the development of the student's ability to work effectively with Cartesian coordinate systems. On the surface it appears to be the most revolutionary chapter in the text, but in practice it has been one of the easiest and most rewarding to teach.

Since no rational number satisfies the equation $x^2=2$, a simple algebraic extension of the rationals is made by postulating the existence of a root for this equation and adjoining it to the rationals. For the fourth time we state our fundamental postulates and assume that they govern the operations of addition and multiplication in the set of numbers of the form $a+b\sqrt{2}$ with a and b rational.

At this stage the better students are able to present, with excellent understanding, proofs of the following chain of theorems:

1. If a and b are rational numbers, then $a+b\sqrt{2}=0$ if and only if $a=0$ and $b=0$.
2. $a+b\sqrt{2}=c+d\sqrt{2}$ if and only if $a=c$ and $b=d$.
3. The product of two numbers, $a+b\sqrt{2}$ and $c+d\sqrt{2}$, is zero if and only if one of the numbers is zero.
4. There are exactly two numbers of the form $a+b\sqrt{2}$ whose squares are 2.
5. Division in this new set of numbers (except by 0) is always possible and is unique.
6. Every number $a+b\sqrt{2}$ is a root of a quadratic equation with rational coefficients.
7. There is no number in our set whose square is 3.

The final extension of our number system occurs when the reals are presented as infinite decimals. This is the fifth set of numbers that has been systematically studied during the year. For each set the one basic group of postulates has superimposed an algebraic structure upon the set. This pattern of repetitious formulation of the postulates makes it crystal clear to all students who have the capacity for further mathematical study that the structure of a mathematical system unfolds from the postulates that characterize it.

The geometry text

The geometry is primarily plane geometry. A summary of space geometry is presented in one chapter and a brief introduction to coordinate geometry in a second, but these chapters are out of the main line of the development. The geometry can best be described as *Euclid made precise and rigorous according to the standards of modern mathematics*. It is well known that around the turn of the century David Hilbert formulated the first mathematically acceptable set of postulates for geometry. The Ball State geometry is based upon a modified version of the Hilbert postulates. The selection of the postulate set and its formulation in language acceptable to mathematically immature students constituted the basic problem. Once this was done the elaboration of the geometry required only time for its completion.

It is not generally recognized, even in the geometry classroom, that there are two kinds of geometries. The conventional geometry class makes little or no distinc-

tion between these and moves freely back and forth between

1. Physical Geometry
2. Mathematical Geometry.

Euclid himself does not differentiate between these two geometries. Such language as:

Draw a line.
Measure an angle.
Place point A on point B.
Rotate the line segment.
Point A moves along the line.

is the language of physical geometry. The words *draw*, *measure*, *place*, *rotate*, *move*, etc., as used in the ordinary geometry classroom, refer to physical activity and not to mathematics. These terms are not defined in traditional geometry textbooks and their significance is not made clear by postulates.

On the other hand, when teachers tell students that the assertions they make in their geometric proofs must be justified by appealing to postulates and previously proved theorems, and not by pointing at physical objects (like figures drawn on the board), then they are talking about mathematical geometry.

The Ball State geometry distinguishes sharply between these two geometries and establishes from the beginning that the purpose of the course is to study mathematical geometry. Repeatedly it is observed that points and lines are things that exist only in our minds. Pencil marks and chalk marks are physical objects used to picture lines and points. The student is told:

Draw a *picture* of a line.
Mark a *picture* of a point.
Measure a *picture* of an angle.

You will observe that the postulates listed below use none of the words that refer to physical activity—draw, mark, extend, move, etc.

The postulate set employed in the Ball State geometry includes a total of eighteen postulates. Of course not all of these are logically necessary. Several could be proved as theorems from the others, but in general these proofs are difficult.

The postulates are presented as

1. Postulates of incidence (three postulates)
2. Postulates of order or betweenness (five postulates)
3. Postulates of congruence (seven postulates)
4. The Postulate of Archimedes
5. The completeness postulate
6. The parallel postulate.

In the additional chapter on space geometry there are four more postulates which suffice to prove the theorems of three space.

It is emphasized that the geometrical relations of *congruence* and *betweenness* are *undefined* relations. Incidence is interpreted in terms of set membership. That is, the statement, "Point P *lies on* line l" means that the point P is one of the points in the set of points constituting the line l. The language of set theory is used consistently. For example, it is remarked that the Euclidean plane is a *set of points* and that the objects of geometric study are certain subsets of this large set which we refer to as lines, segments, triangles, circles, etc. In other words, *every geometric figure is a set of points.*

The postulates for the geometry are given below. Only the postulates for plane geometry are given.

The Incidence Postulates

1. There are at least three points not all on a line.
2. For any two different points, there is exactly one line containing these points.
3. Every line contains at least two points.

The Betweenness Postulates

4. If B is between A and C, then A, B, C are three different, or distinct, points on a line.
5. For every three points on a line, exactly one of them is between the other two.
6. Any four points on a line may be named A_1, A_2, A_3, A_4 so that the only betweenness relations are the same as the order of the subscript numbers. That is, the betweenness relations are A_2 between A_1 and A_3, A_2 between A_1 and A_4, etc.

7. If A and B are two points, then there is at least one point C such that B is between A and C, and at least one point D such that D is between A and B.
8. Every line l separates the plane. By this we mean that all points of the plane not on the line are divided into two sets, called the two sides of the line, having the following properties:
   a) If P and Q belong to one of these sets then no point of l is between P and Q. We say then that P and Q are on the same side of l.
   b) If P and Q are in different sets, then there is a point on l which is between P and Q.

Linear congruence

9. Given points A and B on a line l and a point A' on a line l', then, on a given side of A' on l', there is exactly one point B' such that AB is congruent to $A'B'$. We write this as $AB \cong A'B'$, and read this as "AB is congruent to $A'B'$."

   In congruence the order of the points does not matter, that is, if $AB \cong A'B'$ then also $AB \cong B'A'$, $BA \cong A'B'$, and $BA \cong B'A'$.
10. Congruence also satisfies the following:
    a) $AB \cong AB$. (A segment is congruent to itself.)
    b) If $AB \cong A'B'$ then $A'B' \cong AB$.
    c) If $AB \cong A'B'$ and $A'B' \cong A''B''$, then $AB \cong A''B''$.
11. (This postulate tells how to "add" and "subtract" segments.) Suppose that B is between A and C on a line l, and that B' is between A' and C' on a line l'.
    a) If $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.
    b) If $AC \cong A'C'$ and $BC \cong B'C'$, then $AB \cong A'B'$.

Archimedes' Postulate

12. If AB is any segment on a line l and CD any other segment, then there is a finite number n of points $A_1, A_2, \cdots, A_n$ on l such that the points $A, A_1, A_2, \cdots, A_n$ are arranged in this order, all the segments $AA_1, A_1A_2, \cdots, A_{n-1}A_n$ are congruent to CD, and either $B = A_n$ or B lies between A_{n-1} and A_n.

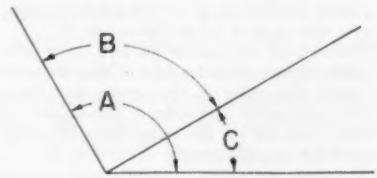
The completeness postulate

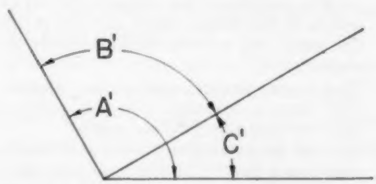
13. For every line l and every point A on l, and for every positive real number x, there is, on a given side of A, a point B such that $\overline{AB} = x$. (Note: The symbol "$\overline{AB}$" represents the length of the segment.)

Congruence postulates for angles

14. If $\angle A$ is not a straight angle and if r' is a ray from A' on a line l', then, on a given side of l', there is exactly one ray s' from A' such that the angle A', with sides r' and s', is congruent to angle A. We write this as $\angle A' \cong \angle A$.

    If $\angle A$ is a straight angle, then the straight angle at A' with one side r' is the only angle with one side r' congruent to angle A.
15. a) $\angle A \cong \angle A$.
    b) If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.
    c) If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.
16. If, in the figure below, any two of the pairs $\angle A$ and $\angle A'$, $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$ are congruent, then the third pair of angles are congruent. In other words:
    a) If $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$, then $\angle C \cong \angle C'$.
    b) If $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$, then $\angle A \cong \angle A'$.
    c) If $\angle C \cong \angle C'$ and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$.





The angle A may be a straight angle.

17. If for $\triangle ABC$ and $\triangle A'B'C'$, $AB \cong A'B'$, $AC \cong A'C'$, and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$.

The parallel postulate

18. Through a point not on a line there is no more than one line parallel to the given line.

Below are listed some of the definitions approximately as they are stated in the text. In conventional texts most of the definitions given here are either omitted or are replaced by nonmathematical descriptions which appeal to the students' intuition alone.

Some definitions

1. A *geometric figure* is any set of points in the plane.
2. If A and B are any two points on a line, then the set of points consisting of A and B and all the points between A and B will be

called the *line segment* (or segment) determined by A and B. The segment will be indicated by AB. The points A and B are called the ends of the segment.

3. If A, B, C are three points not all on a line, the set of points of the segments AB, BC, AC is called a *triangle*. We call the points A, B, C *vertices*, and the segments AB, AC, BC, *sides*. The triangle will be denoted by $\triangle ABC$. We read the symbol $\triangle ABC$ as "triangle ABC."
4. If O is a point of a line l and A and B are two other points of l, then we say that A and B are on the *same side* of O if O is not between A and B. We say that A and B are on *opposite sides* of O if O is between A and B.
5. A *ray* from O, or a ray with end O, is a set of points consisting of O and all the points on one side of O of a line l through O.
6. The set of all points on two rays from a point O is called an *angle*. The point O is called the *vertex* of the angle, and the two rays are called the *sides* of the angle. If the two rays are on the same line, the angle is called a *straight angle*.
7. Consider an angle, $\angle AOB$, not a straight angle. The *interior* of $\angle AOB$ is the set of all points that are both on A's side of line OB and on B's side of line OA.

   The *exterior* of the angle is the set of all points in the plane which are not points of the angle and are not in the interior of the angle.
8. Two angles are *adjacent* to one another if they have a common side and vertex and have no common interior points.
9. Let AB be any segment and $A'B'$ any segment on a line l'. On the same side of A' as B', let B'' be the point such that $AB \cong A'B''$. If B' is between A' and B'' we say that $AB > A'B'$. If B'' is between A' and B' we say that $AB < A'B'$.
10. Two triangles are said to be *congruent* if they can be labeled $\triangle ABC$ and $\triangle A'B'C'$ such that $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$. We then write $\triangle ABC \cong \triangle A'B'C'$ and read this symbol as "triangle ABC is congruent to triangle $A'B'C'$."
11. Let $P_1, P_2, \cdots, P_n$ be any set of n points. The set of points on all the segments P_1P_2, $P_2P_3, \cdots, P_{n-1}P_n$ is called a *polygonal path*. The path is said to join or connect the end points P_1 and P_n. The points $P_1, \cdots, P_n$ are called the *vertices*. In particular if $P_1 = P_n$ the polygonal path is called a *polygon* and designated as polygon $P_1P_2, \cdots, P_{n-1}$.
12. A *circle* is determined by a point O, called the center, and a positive number r. The circle is the set of all points A of the plane such that the distance $\overline{OA} = r$. If A is a point of the circle, then the segment OA is called a *radius* (plural radii). If A and B are on the circle, and the segment AB contains O, then AB is called a *diameter*.
13. If a circle has a center O and radius r, the *interior* of the circle is the set of all points B such that $\overline{OB} < r$. The *exterior* of the circle is the set of all points C such that $\overline{OC} > r$.
14. Suppose that with each positive integer n there is associated a number x_n. The numbers $x_1, x_2, \cdots, x_n, \cdots$ then form an *infinite sequence*. The number x_n is called the nth term of the sequence.
15. The number L is called the *limit* of the sequence x_n if, for large values of n, x_n is necessarily arbitrarily close to L. We then say that L is the limit of the sequence of terms x_n, and write $L = \lim x_n$.
16. The *circumference* C of a circle is the limit of the perimeters, P_n, of regular inscribed polygons, $C = \lim p_n$.
17. The number C/d, which is the same for all circles, will be denoted by the Greek letter π (pi).
18. The *area* S of a circle is the limit of the areas of inscribed regular polygons. That is, $S = \lim S_n$.
19. Given a circular arc, let l_n be the length of a regular polygonal path of 2^n sides inscribed in the arc. The *length* l of the arc is defined to be the following limit: $l = \text{limit } l_n$.
20. On a circle of radius r an arc of length $1/90 \cdot \pi r/2$ is subtended by a central angle of *one degree* ($1°$).
21. If on a circle of radius r a central angle subtends an arc of length L, then the *measure* of the angle in degrees is $(L/\pi r) \cdot 180°$.

The theorems listed below are included for the most part because they are either postulated or ignored in conventional texts. By "ignored" it is meant that the truth of the theorem is assumed upon the basis of intuitive evidence presented by physical objects—perhaps without the author's realization that he is making this tacit assumption.

As an example of this, Euclid himself assumes without comment in the first proposition of Book 1 of *The Elements* that if AB is any segment, the circle with center at A and radius AB intersects the circle whose center is B and radius BA. There is no evidence that Euclid was conscious of the need for a geometric assumption here. He was deceived by his drawings. Another serious omission in the original Euclidean postulates resides in the omission of postulates of order, relating to the relation of *betweenness*.

Of course most of the theorems in the Ball State geometry are conventional, describing familiar properties of triangles, circles, parallelograms, etc.

Some theorems

1. Two different lines intersect in, at most, one point.
2. There are at least three lines in the plane.
3. Let O be any point of a line l. The point O separates the line into two sets, called the two sides of the point O on the line, having the following properties:
   a) If P and Q belong to the same side of O, then either P is between O and Q, or Q is between O and P.
   b) If P and Q belong to different sides of O, then O is between P and Q.
4. If P and Q are both in the interior of $\angle O$, then the segment PQ is in the interior of $\angle O$.
5. A ray r, from O, other than r_1 or r_2, lies except for its endpoint O either completely in the interior or completely in the exterior of $\angle O$.
6. If P is in the interior of $\angle O$ and Q is in the exterior of $\angle O$, then the segment PQ contains a point of the angle.
7. If AB and $A'B'$ are any two segments, then $AB > A'B'$, or $AB \cong A'B'$, or $AB < A'B'$, and exactly one of these holds.
8. If $AB > A'B'$, then $A'B' < AB$, and conversely.
9. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
10. If $AB \cong A'B'$, then $\overline{AB} = \overline{A'B'}$.
11. If $\overline{AB} = \overline{A'B'}$, then $AB \cong A'B'$.
12. For any two angles $\angle A$ and $\angle B$, $\angle A < B$, $\angle A \cong \angle B$, or $\angle A > \angle B$, and exactly one of these is true.
13. All right angles are congruent to one another.
14. If l is any line and A any point not on l, then there exists at least one line on A parallel to l.
15. No matter what unit of length is chosen, the ratio of two segments is the same number.
16. Suppose that circles with centers A and B each have radius $r > \frac{1}{2}\ \overline{AB}$. Then these circles intersect in two points on opposite sides of the line AB.
17. Exactly one circle is circumscribed about a regular polygon.
18. Two regular polygons are similar if and only if they have the same number of sides.
19. For a given circle let p_n be the perimeter of a regular polygon of n sides inscribed in the circle. The numbers p_n, $n = 3, 4, 5, \cdots$, form a sequence which has a limit.
20. The ratio C/d (circumference to diameter) is the same for all circles.
21. If two central angles are congruent, they subtend arcs of equal length.

Conclusion

When we began our experimentation we were fully conscious that a mathematical program which places emphasis upon logical reasoning and presents the austere sequence of postulate, definition, and proof may discourage many students from continuing their study of mathematics. We rather expected an increase in the drop-out rate both at the end of the first year of algebra and at the end of the geometry. However, we felt that from here on drop-outs should be far fewer than in the past and that as a net result more students would proceed successfully to the study of advanced mathematics. Every teacher of college mathematics knows how many high school students flounder pathetically through three or four years of high school mathematics and come to college with almost no chance for success in college mathematics.

Rather surprisingly, at this time there is no indication that the drop-out rate has increased because of our experimental program. We cannot yet judge the significance of this fact. Certainly we find in our third- and fourth-year mathematics courses students who, by our standards, should not be there. We look forward with particular interest to our students' performance at the college level.

It seems to be no more difficult to teach a bright fourteen-year-old student in grade 9 what it means to make a correct mathematical proof than it is to teach a bright twenty-year-old college junior. In some respects the task is easier with the fourteen-year-old. In general, the teachers who have taught in our experimental program have faced greater problems of adjustment than have the students. Over the years a most unfortunate mathematical language has developed and has been propagated by our texts. This is a language that refers primarily to the *symbols* of mathematics and seldom to the concepts. It is a language that has been found useful in persuading students to perform mechanically certain approved manipulations of symbols. Examples of such language are:

Similar terms
Literal numbers
Evaluation is the process of *putting* a number in place of a letter in an algebraic expression.
A number *preceded by* a plus sign is called a positive number.
The absolute value of a number *remains* when the sign is *dropped*.

When adding a column of algebraic terms we may add *upward* and then check by adding *downward*.

In subtracting and multiplying polynomials the *vertical arrangement* is best.

Any parenthesis without a coefficient really has a coefficient of 1.

To divide a polynomial by a monomial divide each term by the monomial *uniting* the resulting quotients by their proper signs.

A term may be *dropped* from either member of an equation provided the corresponding opposite term is *written* in the other member.

To square a monomial, square the numerical coefficient and *give* to each literal factor an exponent twice as large as the one *appearing* in the original monomial.

Write the square root of each of the two squares and *connect* the results by the sign of the remaining term.

Write the product of the numerators over the product of the denominators, *removing* all factors that are common to the two products.

The value of a fraction is not altered if we *change* any two of the three signs associated with the fraction.

This language of *put, take, write, connect, unite, drop, remove, change, transpose, cancel, group, combine, arrange vertically, check by adding down*, etc., is the sort that one would use in instructing someone of limited intelligence how to perform a purely mechanical task. There is no indication that concepts are under consideration. The principal difficulty that teachers who have worked with us have faced is that our material is *concept-centered* rather than *symbol-centered*. It is harder to talk to students about *ideas* than it is to tell them how to arrange marks on paper, but in the long run it leaves one with a better taste in his mouth.

Have you read?

Ault, Colonel John W. "Carry-Over," *Mathematics Magazine*, May–June, 1959, p. 264.

Do you have time to read one page? If so, read this article and think about the contribution of mathematics in the field of problem-solving. The author says students are inclined to leave "Given," "To Find" and "Solution" behind at the conclusion of their mathematics course. How can one teach so that this doesn't happen? It is simple, according to the author—the teacher applies the problem-solving method to all problems he meets in the classroom and out. Read the article and then try to follow its advice.—*Philip Peak, Indiana University, Bloomington, Indiana.*

Boyer, C. B. "Descartes and the Geometrization of Algebra," *The American Mathematical Monthly*, May, 1959, pp. 390–393.

Today we hear much about the integration, reorganization, or tying together of all mathematics. The author of this article shows that such activity is not new to mathematics.

Your students will know of Descartes. They will be interested to know that he thought geometry had too many diagrams which fatigue the imagination, while he considered algebra to be an obscure art which embarrassed the mind. Therefore, he proposed to free geometry of diagrams through algebraic procedure and to give meaning to algebra through geometric interpretation. A great and radical step in his day, this was the beginning of the great reform in mathematics.

Descartes was convinced that all mathematical sciences proceed from the same basic principles. Using lines to represent all magnitudes enabled Descartes to transcend three dimensions and brought in the idea of unlimited extension. Your advanced students will read this and begin to see the structure of mathematics.—*Philip Peak, Indiana University, Bloomington, Indiana.*

Some observations on mathematics curriculum trends

REV. D. B. SMITH, *Saint Louis Priory, Creve Coeur, Missouri.*

Is modern abstract algebra a suitable subject for secondary school students? Is calculus a suitable subject for secondary school students?

AT THIS TIME, when the mathematical education of the youth of the country is being subjected to such close scrutiny and is the object of so many recommendations, it may be of interest if yet one more point of view is put forward. The writer is only in his second year of teaching in this country and has had only this period in which to find his bearings in the educational world here. His views can therefore claim no respect from the more wise, but perhaps for this very reason it may be of interest to read of one teacher's first impressions of the present mathematical situation and of the suggested improvements in the high school program for college preparatory mathematics. If the judgments tentatively put forward in this article are found to be invalid, they do at least spring from what may be an unusual viewpoint, that of one who has but recently come to this country, and it may be well that the situation should be seen and discussed from as many points of view as possible.

Before discussing the high school curriculum, it is perhaps worth putting on record without comment some of the early impressions of one who had been used to teaching mathematics in another educational framework. There was, first, a sense of relief in the discovery that secondary education was not fettered by the tyranny of an examination system which in England is always in danger of becoming an end rather than a means. It was reassuring, too, to find that institutions of higher learning were determined not to dictate to the high schools, but to leave them as free as possible to follow what seemed the best course for the individual school. To be sure, this freedom from standardization has its own dangers and drawbacks, notably that the colleges cannot presume any uniform standard of attainment in their freshmen, so that there must be the danger of a good deal of repetition in first-year college work. Then there were some early surprises concerning the traditional mathematical program.

First, it seemed strange to find that a whole year was devoted to one branch of mathematics to the exclusion of all others, and then another year to a second branch, and so on; instead of teaching them concurrently, so that no boy would have to pick up the threads of algebra in his junior year, having done none for fifteen months. A second surprise was to find that geometry was apparently still being widely taught directly from the *Elements*, even to the use of archaic translations, a practice which I had thought had been generally abandoned toward the end of the nineteenth century. One other point may be mentioned: college textbooks on analytic geometry seem to be concerned mostly with conics having equations with numeri-

cal rather than literal coefficients, and the bulk of the examples deal with the rather mechanical problem of finding the foci, axes, etc., of such conics, or the inverse problem, rather than with proving results using polar and other general properties of conics, which require some geometrical insight.

The foregoing remarks are not strictly relevant to the general purpose of this article, and there are of course many other differences of detail between American and British teaching of mathematics. These points have simply been made by way of example. Since the suggestions to be put forward in the remainder of this article are not based on any authority, it may be best to be quite frank in the use of the first person. I may say, therefore, that I feel some concern about present trends in the reform of the high school mathematics program. It seems to me that there is a real danger that a heavy emphasis on logical formalism may have a stultifying effect on the immature student. "Every man is a Microcosm," wrote Sir Thomas Browne; and in a somewhat similar way the mathematical development of every student may be seen as, to some extent, a repetition in miniature of the historical development of mathematics.

The parallel cannot of course be pushed too far. Many topics which received much attention in the past are no longer of much interest or profit, and have rightly been discarded. Nor would it be sensible to insist on early methods when later ones have proved to be of greater ease and power. It would, for instance, be absurd nowadays to make a first approach to the study of the conics by way of the works of Apollonius.

I am convinced, nevertheless, from my teaching experience that it is a mistake in the early stages of any branch of the subject to become too involved in the logical foundations. Not that one should ever allow students to do slipshod work, nor, for example, allow them to be satisfied with a misty vagueness about infinitesimals, those "ghosts of departed quantities" at which Bishop Berkeley sneered. No; one must of course always insist on a degree of accuracy consistent with the student's development, as well as on a sense of style. But I do feel that too great a zeal about the logical basis is misplaced in the early stages of a student's mathematical development. After all, the thorough investigation of the foundations of mathematics is historically a late development of the last hundred years; and it is in this sense that I feel the individual's progress must be, to some extent, parallel to that of the subject itself. Intuition and logic—using these words in a "popular" and inexact way—both have their part to play, and the latter, in its rigorous form, requires the greater maturity of mind.

An example will illustrate this. In the overspecialized English educational system, high school students with marked mathematical ability have the opportunity of studying relatively "advanced" topics. When I first began to teach projective geometry from the synthetic point of view, my inexperienced enthusiasm was not satisfied with the standard school textbook then in use, in which the subject was introduced on a metrical basis (metrical definition of cross-ratio, etc.). I tried to put before the students a truly nonmetrical axiomatic projective geometry, but I soon realized that this was a mistake. Boys of sixteen or seventeen years of age have not the maturity for this sort of approach. Paradoxically, they become bewildered by the very simplicity of it. On the other hand, if one is prepared to allow metrical considerations in the introduction to the course, one can very soon get away from explicit reference to them as progress is made, and these same boys are then capable of producing work on projectivities, involutions, and the projective conic, which has all the elegance that one associates with pure projective geometry. When they go to the university it is time enough for them to turn back to re-examine the fundamental assumptions. In this way the

individual's development follows the historical development of the subject in its broad outlines.

This example of projective geometry is of course irrelevant in the context of the American high school, but it does perhaps illustrate the point which I feel rather strongly. In recent times the influence of the mathematical logician has become very strong, and I think that too great an emphasis on formalism in the early stages of mathematical development could be a real danger in hampering the spirit of ingenuity and enjoyment which should be a driving force for the young student of mathematics. I repeat that this is no plea to allow slovenly proofs, much less any meaningless "learning by rule," but I do think that excessive formalism may stifle the power of insight and initiative, and lead to boredom.

There now appears to be a welcome trend away from such excessive formalism in the teaching of elementary geometry, but I wonder whether there is not a danger that it will be transferred to the teaching of algebra. With Professor Sawyer,* I doubt that the effort to be "modern" is a solution to all problems. My feeling is: let us encourage the high school student to get on, to make progress in building up his mathematics. The proper moment for a really close scrutiny of the foundations of this edifice is at the university, when greater maturity has been achieved. The outstanding impression made on the newcomer here is of the students' need to learn initiative and not to have to be told what is the next step to be taken, and I am not persuaded that in some of the recommendations put forward the high school student would be capable of doing anything more than *follow* the teacher.

I know from experience that able and well-prepared ninth-graders are capable of beginning calculus, and *not* as a meaningless mechanical procedure, but of studying it with a reasonable degree of accurate statement, which of course can be refined as they progress through high school. In their later high school careers—I am speaking of those with marked mathematical ability—they are capable of relatively advanced work in analysis and algebra, as well as in various methods of geometry, both pure and analytic, such as inversion, elementary synthetic projective geometry, and the use of general homogeneous coordinates. I am not suggesting that in a well-balanced educational program there would be time to study all these mathematical topics. My point is rather that, whereas the superior student of high school age is capable of making great progress, I doubt that he is ready for a heavy emphasis on what may seem to him the arid logical foundations of any branch of the subject. I am skeptical that such a student would be enthusiastic about the need to take a dozen steps in proving that, if $a=b$, then $-a=-b$.

It seems to me that the search for structure is something that should begin rather late in the mathematical development of the individual, just as it did in the history of the subject. It seems to make better sense to wait until the young mathematician has had considerable experience in a number of branches of the subject before he is put to studying unifying principles which will then enable him to appreciate how the various branches often dovetail into each other, rather than to have him study such principles "in the void" before he can understand what precisely it is that they unify.

I have only a rudimentary acquaintance with group theory, but I can see what exciting possibilities it puts before one who has had experience of, say, matrix algebra, the geometry of Euclidean, affine, and projective transformations, as well as of number theory; prior to such experience, however, might it not seem a very barren abstraction to the immature student? True, it is suggested that geometrical examples of transformation groups be put before the student, but I wonder whether

* THE MATHEMATICS TEACHER, LII (April 1959), p. 272.

his geometrical knowledge is not so slight after only the present tenth grade work as to render such concrete "realizations" of much less value than they would be at a later stage. Mathematics as the study of structure is a late development, and probably it could not have come about much earlier; it seems to me that the individual likewise needs to have made considerable progress in several branches of the subject before he can appreciate the more refined processes of abstraction which lead to a view of mathematics as a network of formal relations.

It may be only lamentable conservatism that makes me doubt whether the early introduction of the language of sets will prove to be a cure for many weaknesses. Perhaps it will be a useful means to understanding if it can really be integrated into the elementary course, but I confess I have seen no successful attempt as yet. Without it, students can still be brought to understand what they are doing and to see something more in their work than the application of meaningless rules, and here of course a true notion of variables is essential—this can be secured without recourse to the somewhat exotic methods sometimes recommended. Fundamentally, the most important thing is to stimulate the initiative of the student, to reply to each of his questions by another question, in the Socratic manner, until he comes to see for himself the solution of his original problem.

What has been written above may seem at once both reactionary and revolutionary: reactionary, because of my skepticism about the suitability of modern abstract algebra as a high school subject; and revolutionary, because of my conviction that calculus can be begun in the eleventh grade at the latest. It may simply be that this dog is too old to learn new tricks; and, indeed, much of what has been written may be a tilting at windmills, since the Report of the Commission on Mathematics does not put such heavy emphasis on the logical foundations as one has read in the suggestions of some other writers. All the same (and of course I am only considering students with considerable mathematical ability), I do feel that calculus is easier and more stimulating to students of high school age than is abstract algebra. As I say, there is no reason for calculus, even at first, to be a mere collection of rules for solving particular types of problems; the student's grasp of the justification for the procedures is gradually deepened as he approaches what may more properly be called analysis. It is difficult to see, incidentally, how a satisfactory physics course can be given without elementary calculus as a tool.

It is true that it would not be possible to make any sudden and drastic change in the program, and the reforms that one has heard proposed have had to be kept reasonably within the present framework. Moreover, it would not be possible to introduce the high school student of superior mathematical ability to the calculus as long as the beginning of algebra is delayed until the ninth grade. An early start is what is needed, but how that can be accomplished with the huge numbers to be taught in the grade schools is another problem.

In the school where I am teaching, we are fortunate in having small divisions, but, on the other hand, the enrollment is not large enough to ensure that all the boys in a "set" are as nearly equal in ability as one would wish. We are now taking boys into the seventh grade, hoping that they may thereby make considerably more progress before graduation in foreign languages, particularly, but also in mathematics. As this is only the fourth year of the school's existence, the mathematics program is still in a very experimental stage. In general, the present plan is to run arithmetic, algebra, and geometry concurrently from seventh through ninth grades. (Ideally, there would be only a very little arithmetic done in the ninth grade, but the teachers both of mathematics and of science feel that the arith-

metic mastered by our present students when they came to us from the eighth grade was insufficient.) The tenth-grade year, it is hoped, will eventually see the more mathematically able boys reaching a standard in algebra, geometry, and trigonometry roughly equivalent to the old traditional four-year high school course in all but a few topics.

There would then remain two years for further study. Clearly a substantial part of this time should be spent on algebra (for instance, complex numbers, inequalities, theory of equations, permutations and combinations, binomial theorem and series, and other elementary series after an introduction to convergence). I would personally be glad if some time could be found to study a little more geometry, because, of all elementary mathematics, geometry is the easiest branch in which to encourage a student's appreciation of a sense of style. (I sometimes wonder whether we are going to have to abandon that side of our work as educators just because statistics is now so important a subject in the modern world.) Inversion, for instance, is a good example of an elementary type of transformation; and one might be able to consider the harmonic properties of the quadrangle, if only as metrical consequences of the theorems of Menelaus and Ceva. Together with the algebraic topics mentioned above in parentheses, the students might have time in these two years to cover the equivalent of a full year's course in calculus and analytic geometry (which, as others have suggested, could be at least introduced a good deal earlier in the course). I would like to see analytic geometry emphatically treated as *geometry*, and not degraded into a sort of superior graphing.

All this is a very personal view, and of course one must be ready to modify one's decisions according to the requirements of the colleges. Moreover, it remains to be seen how far this outline of a program may prove to be overoptimistic. It will certainly remain so as long as the students do not learn greater initiative. Even our present (and first) seniors have not yet learned, despite my fulminations, that they should be extending the frontiers of their mathematical knowledge largely by their own efforts, instead of expecting everything to be "handed to them on a plate." And that basically is one of my fears about the emphasis on modern abstract algebra. Able students may be able to *follow* the teacher, but will they learn independence and initiative in their mathematics?

Annual business meeting

Notice is hereby given, as required by the Bylaws, that the Annual Business Meeting of the National Council of Teachers of Mathematics will be held at the Statler Hilton Hotel, Buffalo, New York, at 4:00 P.M., April 22, 1960.

A problem with touching circles

JOHN SATTERLY, *University of Toronto, Toronto, Canada.*

Those who enjoy geometric construction problems will find much pleasure in drawing combinations of touching circles.

THE FOLLOWING problem is illustrated and discussed in a few books and journals [1, 2].* Given a semicircle A (*Fig.* 1) of radius a and two equal semicircles B and B' with radii $b=a/2$, it is required to construct circles $C, D, E, F \cdots$ as shown in the figure circle C to touch circles A, B, B', circle D to touch circles A, B, C, circle E to touch A, B, D and so on. To economize on letters, let us use capital letters for circles *and* their centers and the corresponding small letters for their radii.

Circle C. Its size and center can be derived easily by elementary geometry, for center C must by symmetry be right above A, and in the right angle triangle

$$ABC, \quad AB=b, \quad BC=b+c,$$
$$AC=a-c=2b-c.$$

The Theorem of Pythagoras gives

$$\overline{BC}^2=\overline{AB}^2+\overline{AC}^2 \therefore (b+c)^2=b^2+(2b-c)^2$$

whence $c=\frac{2}{3}b=\frac{1}{3}a$ and $AC=\frac{2}{3}a$.

GENERAL SOLUTION

Casey in his Sequel to Euclid considers the problem of describing a circle to touch three given circles. The geometrical construction and proof are complicated. It is much easier to use coordinate geometry and the equations of circles. Take A as origin, axis of x along AB, axis of y along AC. As we proceed we shall write X_1, Y_1 for the coordinates of the center of C; X_2, Y_2 for D; X_3, Y_3 for E and so on.

Circle C. The center of C must be at distances $a-c$, $b+c$, $b+c$ from A, B, B' respectively, so it must be at the common intersection of three circles of radii $a-c$, $b+c$, and $b+c$ whose centers are A, B and B' respectively. Imagine arcs of these three circles to be described as in *Figure* 2. We proceed to find X_1, Y_1 and c by solving simultaneously the three equations of the circles:

that from center A, $x^2+y^2=(a-c)^2$

that from center B, $(x-b)^2+y^2=(b+c)^2$

that from center B', $(x+b)^2+y^2=(b+c)^2$.

To make the problem concrete, I shall take $a=6$ units of length—cms. or half-inches are convenient for the draftsman.

The equations of the three circles are now:

that with center A, $x^2+y^2=(6-c)^2$ (1)

Figure 1

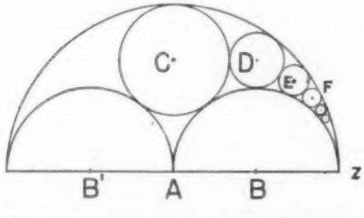
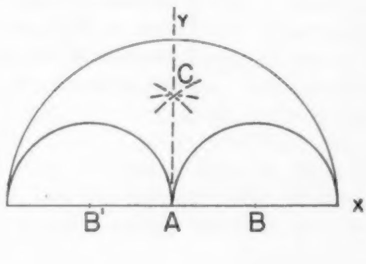


Figure 2



* Numbers in brackets refer to references at the end of the article.

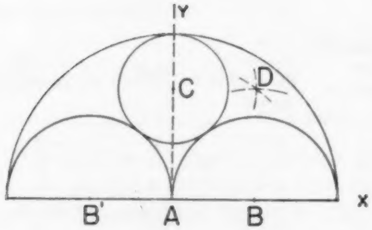


Figure 3

that with center B, $(x-3)^2+y^2$

$$=(3+c)^2 \quad (2)$$

that with center B', $(x+3)^2+y^2$

$$=(3+c)^2 \quad (3)$$

Solve for X, Y, and c. From equation (2) and (3) we get $X_1=0$ (also obvious). On substitution in equation (1) we get $Y^1=(6-c)^2$ and on substitution in equation (2) we get $9+Y_1^2=(3+c)^2$. Therefore $c=2$ units and $Y_1=4$ units.

Circle D. The center of this circle must be at distance d from the circles A, B and C, so imagine arcs described as in *Figure* 3 from A, B and C with radii $6-d$, $3+d$, and $2+d$. The value of d must be found, giving the common intersection D. The equations of the three dotted circles are:

that with center A $x^2+y^2=(6-d)^2$ (4)

that with center B $(x-3)^2+y^2$

$$=(3+d)^2 \quad (5)$$

that with center C $x^2+(y-4)^2$

$$=(2+d)^2 \quad (6)$$

The solution of these equations gives (I omit the algebra) $X_2=6-3d$, $Y_2=6-2d$ and $d=1$ or 3. The value 3 evidently refers to circle B' and the value 1 to circle D. Hence $X_2=3$ and $Y_2=4$ units.

NOTE: It so happens that center D is right above center B, so that $ABDC$ is a rectangle, the "*curious rectangle*" of some writers [2].

Circle E. The center E must be the common intersection of three circles of radii $6-e$, $3+e$, $1+e$ described from centers A, B, D respectively. Therefore, to get e, use the following three equations for these circles:

that from center A, $x^2+y^2=(6-e)^2$ (7)

that from center B, $(x-3)^2+y^2$

$$=(3+e)^2 \quad (8)$$

that from center D, $(x-3)^2+(y-4)^2$

$$=(1+e)^2 \quad (9)$$

Their common solution gives $X_3=6-3e$, $Y_3=3+\frac{1}{2}e$ and $e=6/11$ or 2. The value 2 refers to circle C and the value 6/11 to circle E, whence $X_3=48/11$ and $Y_3=36/11$ units.

Circle F. The method of solution is now established. The three circles required have the equations:

that from center A, $x^2+y^2=(6-f)^2$ (10)

that from center B, $(x-3)^2+y^2$

$$=(3+f)^2 \quad (11)$$

that from center E, $(x-48/11)^2$

$$+(y-36/11)^2=(6/11+f)^2 \quad (12)$$

Their solution gives $X_4=6-3f$, $Y_4=2+2f$ and $f=\frac{1}{3}$ or 1. The value 1 refers to circle D. Therefore, for circle F, $f=\frac{1}{3}$, $X_4=5$ and $Y_4=8/3$ units.

Circles G, H. *etc.* These can be found in the same way.

These results are tabulated in Table I. A few words about the right-hand half of the table will be given later.

The centers B', C, D, $E \cdots$ fall very nearly, but not quite, on the semicircle described on diameter $B'Z$ where Z (*Fig.* 1) is the extreme right of the semicircle B.

On further reference to the published literature (and it is curious that one often looks up the publications after working out the problem), we find:

(1) Casey gives "Diameters of circle

$$C, D, E \cdots = \frac{Y_n}{n}$$

where $n=1, 2, 3 \cdots$ for circles C, D, $E \cdots$." See Table I, column 7 for verification.

TABLE 1

CIRCLE	COORDINATES OF CENTER X	Y	RADIUS R $(a=6)$	NUMBER OF CIRCLE, n	$\frac{a}{(n^2+2)}$ $(a=6)$	$2nR$	$a-3R$ $(a=6)$
B'	-3	0	3	0	6/2	0	-3
C	0	4	2	1	6/3	4	0
D	3	4	1	2	6/6	4	3
E	48/11	36/11	6/11	3	6/11	36/11	48/11
F	5	8/3	1/3	4	6/18	8/3	5
G	16/3	20/9	2/9	5	6/27	20/9	16/3
—	—	—	—	—	—	—	—

(2) A. A. Krishnaswami Ayyanger states, "If three semicircles of radii a, $a/2$, $a/2$ be each in contact with the other two as in *Figure* 1, and circles be drawn so that the first of them touches all the three semicircles, the second touches the first and the two semicircles as in the figure, and so on, then the radius of the nth circle $=a/(n^2+2)$. See Table I, column 6 for verification.

The results of (1) and (2) can be obtained by the use of the theorems of the geometrical process called "inversion." By this use Professor Barnes, one of my colleagues, has obtained (1) and (2) and also that $X_n=a-3R$. See Table I, column 8. If we denote by Z_n the distance of a center from an ordinate at the extreme right of *Figure* 1 we have $Z_n=3R$.

(3) Rollett remarks that Professor F. Soddy [3] (the inventor of the term *isotope* which is so familiar to us today) stated in 1936 that:

"If four circles are mutually tangential

$$\left\{\Sigma\left(\frac{1}{\text{radius}}\right)\right\}^2=2\Sigma\left(\frac{1}{\text{radius}^2}\right)$$

and if one of the circles is touched internally its radius must be taken as negative." The statement was made in a poem called "The Kiss Precise." Proof was not supplied.

Subsequently, in a letter to a professor of mathematics Soddy wrote, "As you know I am no mathematician and my Kiss Precise and The Hexlet which you quote were *tours de force* hammered out by sheer algebra and luck. They depended on the reduction of a biquadratic equation of, I think, 23 terms to a quadratic by a transformation I have never really understood."

As an example of the use of Soddy's Theorem take the circles A, B, B', C of *Figure* 1. Here

$$\left\{\Sigma\left(\frac{1}{\text{radius}}\right)\right\}^2 = \left(\frac{1}{-6}+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}\right)^2=1$$

and

$$\Sigma\left(\frac{1}{\text{radius}^2}\right)=\left(\frac{1}{36}+\frac{1}{9}+\frac{1}{9}+\frac{1}{4}\right)=\frac{1}{2}\cdot$$

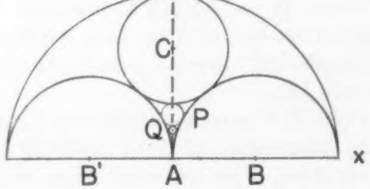
A cognate problem is to fit circles P, Q, R, $\cdots$ (*Fig.* 4), the P circle to touch circles B, B', C, the Q circle to touch circles B, B', P and so on. Obviously the X of all centers equals zero.

As before, to get the P circle we must use equations of the three circles:

that from center B, $(x-3)^2+y^2$

$$=(3+p)^2 \quad (13)$$

Figure 4



that from center B', $(x+3)^2+y^2$

$$=(3+p)^2 \quad (14)$$

that from center C, $x^2+(y-4)^2$

$$=(2+p)^2 \quad (15)$$

The solution of these equations gives $Y_P=2-p$ and $p=4/10$ whence $Y_P=16/10$. Soddy's Theorem and Equation (23) applied to the circles B, B', C also give $p=4/10$.

Proceeding as before for the Q circle we have the three equations:

$$(x-3)^2+y^2=(3+q)^2 \quad (16)$$

$$(x+3)^2+y^2=(3+q)^2 \quad (17)$$

$$x^2+(y-1.6)^2=(0.4+q)^2 \quad (18)$$

and the solution is $q=12/70$ and $Y_Q=72/70$.

We now make Table II for these circles.

We note, that, as shown in column 5, the radii of successive circles are given by the empirical expression, radius $=6/(4n^2-1)$, and that $Y_n=2nR$ as before. These results are, I think, new.

As a further example we calculate the details for a circle V which touches circles B, C, D (*Fig.* 5). The three equations are:

for circle of center B, $(x-3)^2+y^2$

$$=(3+v)^2 \quad (19)$$

for circle of center C, $x^2+(y-4)^2$

$$=(2+v)^2 \quad (20)$$

for circle of center D, $(x-3)^2+(y-4)^2$

$$=(1+v)^2 \quad (21)$$

Their solution gives $X_v=2+v/3$ $Y_v=3+v/2$, and $v=-6$ or $6/23$. The -6 refers to circle A. Circle V therefore has: $X_v=48/23$, $Y_v=72/23$, $v=6/23=0.261\cdots$. We could, of course, have used Soddy's Theorem or Equation (23) for v.

It is worth noting that the incenter of the $\triangle BCD$ is at the point (2, 3) so that the center of circle V does not coincide with this incenter. Only when the three circles touching the inner circle externally and touching each other are equal do these two points coincide.

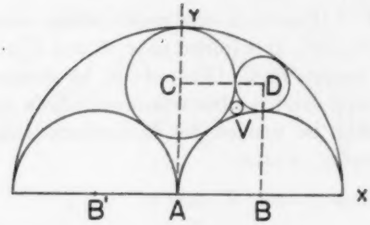


Figure 5

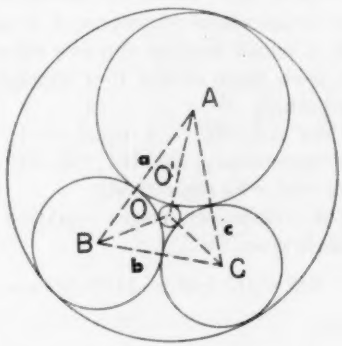


Figure 6

TABLE II

CIRCLE	Y	RADIUS R (Take $a=6$)	n	$\frac{a}{4n^2-1}$ $(a=6)$	$2nR$
C	4	2	1	6/3	4
P	16/10	4/10	2	6/15	24/15
Q	72/70	12/70	3	6/35	36/35
R	16/21	2/21	4	6/63	48/63
S	20/33	2/33	5	6/99	60/99

Proof of Soddy's Theorem

If O (*Fig.* 6) is any point within a triangle ABC, O is joined to A, B and C, and the angles BOC, COA, AOB, be denoted by α, β, and γ, this equation, which can readily be proved by elementary trigonometry, is true.

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma +2\cos\alpha\cos\beta\cos\gamma=0 \qquad (22)$$

In our case (*Fig.* 6) let O be the center of the fourth circle touching externally the three circles whose centers are A, B and C, each of which touches the two others so that each circle of the four touches the three others.

Then AB, BC, CA equal $a+b$, $b+c$, $c+a$ respectively, also OA, OB, OC equal $r+a$, $r+b$, $r+c$ respectively.

The well-known cosine equation of a triangle gives

$$BC^2=OB^2+OC^2-2\,OB\cdot OC\cos\alpha$$

hence

$$\cos\alpha=\frac{(r+b)^2+(r+c)^2-(b+c)^2}{2(r+b)(r+c)} =\frac{r^2+rb+rc-bc}{r^2+rb+rc+bc}.$$

Similarly for $\cos\beta$ and $\cos\gamma$.

The values for the cosines may now be substituted in Equation 22. To simplify, multiply by $\{(r+a)(r+b)(r+c)\}^2$, expand the terms—a tedious piece of work giving expressions of 23 terms in r^6, r^5, r^4, $\cdots$. On addition the terms in r^6, r^5, r^4, r^3 vanish (by a mathematical miracle) leaving only terms in r^2, r and a, b, c which, by a little manipulation, can be thrown into the form given by Soddy's Theorem.

If, instead of using the inside fourth touching circle, we use the outside fourth touching circle of center O' and radius R the corresponding proof to the above leads throughout to the same expressions, but with $-R$ substituted for r. This is because while the r-circle is touched externally by the circles A, B, C the R-circle is touched internally by these circles.

The quadratic equation in the radius obtained near the end of the proof of Soddy's Theorem may, of course, be solved by the ordinary treatment of quadratic equations. It yields:

Radius of the fourth touching circle

$$=\frac{abc}{(ab+bc+ca)\pm 2\{abc(a+b+c)\}^{1/2}} \qquad (23)$$

The second term in the denominator is larger than the first term. If the positive sign before the second term is taken, the radius is r, the radius of the inside touching circle. If the negative sign is taken, the radius is R, the radius of the outside touching circle. R is algebraically negative for the reason given above.

Equation (23) is more convenient for the calculation of r and R than is Soddy's expression. As Soddy, however, points out in his poem "The Kiss Precise," his theorem for touching circles leads on to a similar theorem for touching *spheres*. It may be mentioned here that equation (23) is given as a problem in Hobson's *Trigonometry* [4]. Neither Soddy's expression nor Equation (23) gives the position of the centers of the touching circles, whereas the coordinate geometry treatment above for the problem cited furnishes this information.

Rings of circles

H. G. Forder [5] in his *Geometry*, page 23, says, "If we have two (nonconcentric) circles A and B one inside the other and circles are drawn touching them and one another as in *Figure* 7 (our figure) it may happen that if we continue the ring of touching circles the last one touches the first and the ring closes. *If this happens once it will always happen*, whatever be the position of the first circle" (Steiner).

What conditions must exist between the radii of the two original circles and their positions in order that this may be so?

On making a few drawings I found that the ring would be closed if the outer circle A has a radius of 10 units, the inner circle B a radius of 3 units, and the circum-

TABLE III

Circle	Number	Radius	X	Y	$a-X$
A	—	$a=10$	0	0	—
B	—	$b=3$	3	0	—
C	1	$c=5$	-5	0	$3c$
D	2	$d=.40/11$	10/11	$40\sqrt{3}/11$	$2\frac{1}{2}d$
E	3	$e=40/17$	110/17	$40\sqrt{3}/17$	$1\frac{1}{2}e$
F	4	$f=40/20$	8	0	$1f$

ference of the inner circle passes through the center of the outer circle (*Fig.* 7).

The details for the touching circles C, D (and D'), E (and E'), and F in the ring of circles were worked out as above and are given in Table III where point a is taken as origin, axes as before, and X, Y are the coordinate of their centers.

Always $X=\frac{1}{3}(50-13\times \text{Radius})$ or Radius $=1/13\ (50-3X)$. For circles D and E, $Y=\sqrt{3}\times$ Radius. The relation between n and the radii of C, D, E, F is not obvious. The radii of F, B, C are 2, 3 and 5 which form an a triplet in the Fibonacci series, but I have found no other of these triplets giving closed rings. *Figure* 8 shows, for the same pair of main circles of centers A and B, another set of touching circles bearing out Steiner's remark, that once the chain of circles is found to be closed it does not matter where the chain is begun.

Teachers and lovers of practical geometry will have much pleasure in drawing other combinations of touching circles.

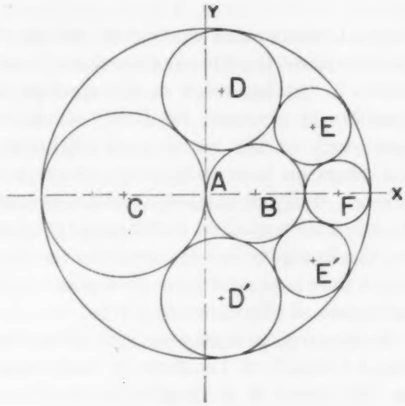


Figure 7

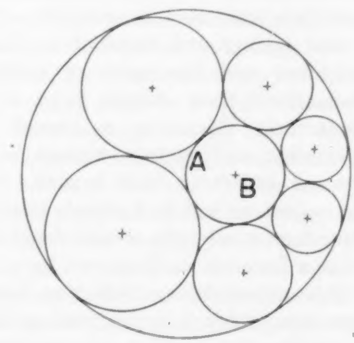


Figure 8

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1. John Casey, *Sequel to Euclid* (3rd ed.; London, Longmans, Green and Company, 1884), p. 103.
2. In the *Mathematical Gazette* (London): A. P. Rollet, XXI (1937), 412, XXII (1938), 287; E. H. Neville, XXII (1938), 288; A. A. Krishnaswami Ayyanger, XXII (1938), 502; J. M. Child, XXXII (1948), 52; Richard Beetham, XXXVII (1953), 53.
3. Frederick Soddy, "The Kiss Precise," *Nature*, CXXXVII (1936), 1021, and CXXXIX (1937), 62.
4. E. W. Hobson, *A Treatise on Plane Trigonometry* (4th ed., Cambridge: Cambridge University Press, 1918), p. 216, ex. 29.
5. H. G. Forder, *Geometry* (London: Hutchinson's University Library, 1950), p. 23, *Fig.* 17.

What is a function?

G. N. WOLLAN, *Purdue University, Lafayette, Indiana.*

The function concept, a concept with many connections to other concepts, needs careful study by teacher and student.

THE MATHEMATICAL concept to which we give the name function is almost as all-pervasive in mathematics as the concept of number. It appears, implicitly at least, very early in the learning of arithmetic and plays an increasingly important role as one's study of mathematics progresses. By the time a student is working on calculus, the function concept occupies the center of the stage, and from that point on it continues to play a leading role.

In the twenty-third yearbook of the National Council of Teachers of Mathematics [1]* there is a chapter on functions written by Professor Rudolph E. Langer of the University of Wisconsin. The purpose of this volume is "to provide reference and background material for both the content and the spirit of modern mathematics." Each chapter is an independent unit discussing a branch of mathematics, and each is written by a recognized expert in that branch. One might expect to find in Langer's chapter on functions a carefully stated definition of what a function is. However, he gives very little attention to a definition, merely remarking, more or less in passing, that a function is a relation between variables. He points out that functions may be represented in several different ways, and he cites a few of these representations as being familiar: viz., the exponential function, the logarithm function, the trigonometric function, and functions which arise from the analysis of vibrations. It seems to me that there is much to be said for this approach to a discussion of "what is a function," a discussion addressed to elementary and secondary school mathematics teachers. I think this chapter is both interesting and rewarding reading.

At least equally interesting and rewarding, however, is Professor Karl Menger's book, *Calculus, A Modern Approach* [2], which is intended for a similar class of readers, and in which there is a somewhat different kind of discussion of this same question. In a section entitled, "The Concept of Function," Professor Menger cites eight illustrative examples beginning with this one: "If, with any number, one pairs 16 times the square of the number (e.g., with 0 the number 0, with 1 the number 16, with 1.5 the number 36), then mathematicians say that a function has been defined. More precisely, the class of the pairs of numbers defined is called a function." Then he writes: "With the preceding example in mind, the reader can easily understand the following general definition: If A is a class of numbers, then a function with the domain A is a class of pairs of numbers such that each number of A is the first element of exactly one pair of the class." He then gives some more examples including some classes of pairs of numbers, which are not functions according to this definition.

In *Universal Mathematics, Part 1* [3], which was written by a small group of mathematicians in 1954 from an outline prepared by the Committee on the Undergraduate Program of the Mathematical Association of America, there is a discussion of the function concept quite similar to Professor Menger's. Here function is defined as follows: "A function is a system consisting of two sets, X and Y, and a rule

* Numbers within brackets refer to references at the end of the article.

of correspondence which assigns to each value of x in X a unique value of y in Y. The set X is called the *domain*, and the set of values of the function in Y is called the *range*." A more formal statement of this definition is also included which defines a function as a triple $\{f, A, B\}$, A and B being sets and f being a relation on, i.e., a subset of, the product set $A \times B$, this latter set being the set of all ordered pairs (a, b) where a is a member of the set A and b is a member of the set B. Here, as in Professor Menger's book, several examples are given to illustrate the definition.

At Princeton University during the past few years the mathematics department has been trying to do a more effective job of interesting the freshmen in mathematics. As part of this effort they chose Professor Emil Artin to teach a special honors course to a class of freshmen selected as being especially good mathematical prospects. This course has resulted in a small paperbound book, *Calculus and Analytic Geometry*, by Emil Artin [4], which is available free from the Mathematical Association of America. It was mentioned earlier that the concept of function is fundamental in calculus, and indeed there are few pages in this little book on which the word function does not appear several times. Yet Professor Artin almost completely sidesteps the question of what a function is. There is only this sentence, included in the discussion of the derivative of a function: "Thus the slope is a *function* of x, i.e., there is a *rule* (perhaps quite complicated) by which we can assign to each number x of a certain collection (not necessarily *all* x) a *single* number $S(x)$, which is in this case the slope of the curve at the given value of x."

In a justly famous and still much used book, *The Theory of Functions of a Real Variable*, by the English mathematician of a generation ago, E. W. Hobson [5], there is a very readable discussion of the function concept in which one finds the following assertion: "It thus appears that an adequate definition of a function for a continuous interval (a, b) must take the form first given to it by Dirichlet, viz., that y is a single-valued function of the variable x, in the continuous interval (a, b), when a definite value of y corresponds to each value of x such that $a \leq x \leq b$, no matter in what form this correspondence is specified." Hobson's discussion includes generalizations of this definition in several ways, e.g., to include more general sets of points than an interval (a, b) and to include multivalued functions.

The above illustrations show that mathematicians do in fact regard functions as being what at least appear to be different kinds of mathematical objects, that is, a relation between variables, a class or set of pairs of numbers, a system or set whose elements are two sets and a relation on, i.e., a subset of, the cross product of the two sets, a rule, and a variable.

Consider the symbol $2x+3$ which in high school algebra we would call an algebraic expression. It is customary for us to ask the high school student of algebra to consider the symbol x to represent any number. Then we ask him to consider the symbol $2x+3$ to be a statement in mathematical language which has the same meaning as "multiply the number x by 2 and add 3 to the result." This last statement is also in mathematical language in the sense that "multiply" and "add" have technical mathematical meanings. The symbol x as here used is an example of a variable in the sense in which this term is commonly used in mathematics. Any other identifiable symbol, such as another letter a or f, or such as the sign *, if used to represent any one of a set of numbers, would also be a variable. Under this interpretation $2x+3$ is an example of a complete sentence of a particular category or class or set to which grammarians attach the name "imperative." If our students of algebra were thoroughly familiar with English grammar, this might be a useful relationship to exploit in teaching algebra. In fact, however, its use would probably

turn out to be of greater benefit to the English teacher than to the mathematics teacher. It is nevertheless probably desirable for us to call it to the attention of our students.

We also do in practice ask the student to think of the symbol $2x+3$ as a variable, i.e., as a symbol representing any one of the numbers which would be obtained as the result of carrying out the instruction or rule $2x+3$. When we use the symbol $2x+3$, we do not attach a label instructing the reader which one of these two interpretations he should make. We demand that he decide from the context which interpretation is called for. Actually, the situation is still more complicated, since we use this same symbol in yet other senses even at the high school level.

For example, we often find it irresistibly convenient to refer to the number x or to the number $2x+3$ as if the symbol x and the symbol $2x+3$ actually were numbers, instead of merely symbols which represent numbers.

More significant, perhaps, is the incontrovertible fact that we ask students to be able to interpret $2x+3$ as a function when the need arises. The symbol $2x+3$ represents a relation between variables of the kind Professor Langer refers to, in the sense that $2x+3$ is one variable, x is another, and the instruction $2x+3$ determines the value of the variable $2x+3$ for each value of the variable x. This same symbol is also interpretable as representing the set of all number pairs $(x, 2x+3)$, and we do in practice ask the student to give it this interpretation when we talk about the graph of the function as the set of points $(x, 2x+3)$. It seems not to be common practice to ask the high school student or the freshman or sophomore in college to interpret $2x+3$ as representing a system whose elements are two sets, A and B, and a third set which is a subset of $A\times B$ of a particular kind which we call a relation. This of course doesn't imply that we shouldn't ask for this interpretation. For the function $2x+3$, the set A is the domain of the function (the set of values of x), the set B is the range of the function (the set of values of $2x+3$), and the relation on $A\times B$ is the set of number pairs $(x, 2x+3)$ which is a subset of the set of all number pairs (a, b) in which a is a member of A and b is a member of B. If f represents this relation on $A\times B$, then $2x+3$ represents the function $\{f, A, B\}$.

To make things a trifle more confusing, just as we cannot in practice resist the temptation to refer to (and to think of) 5 as being a number instead of being a symbol representing a number, so we refer to $2x+3$ as being a function. Actually either interpretation of 5 and also of $2x+3$ is mathematically sound, but it may nevertheless be worthwhile to note the logical distinction even though in practice we tend to ignore it.

It is also common practice to refer to the equation $y=2x+3$ as "the function $y=2x+3$" or as "the functional relation $y=2x+3$." One often also encounters "the function y defined by $y=2x+3$." It is probably clear that one can interpret the equation $y=2x+3$ as representing the same function which we interpreted $2x+3$ to represent. It may not be quite so clear just what this same function is.

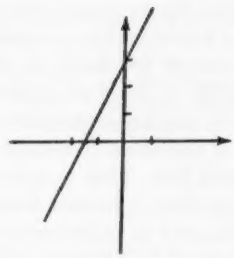
The symbols $(6x+9)/3$, $2x+3$, $2x-y+3=0$ and $y=2x+3$ are all distinguishably different from one another. We customarily say these are different representations of the same function. Most of us would probably say these are trivially different, and especially would we be likely to consider $(6x+9)/3$ and $2x+3$ to be trivially different and similarly $2x-y+3=0$ and $y=2x+3$. However, $(6x+9)/3$ is certainly a different set of instructions or rules than $2x+3$, and the difference between $2x-y+3=0$ and $y=2x+3$ is sufficiently nontrivial that mathematicians have given special names to the two types of representation of function to which they belong. The equation $2x-y+3=0$ is said to be an "implicit" representation and $y=2x+3$ is an "explicit" representation. The equation

$$y^3-2xy^2+x^2y-2x^3-3x^2$$
$$-3y^2+3y-6x-9=0$$

is less obviously an implicit representation of the same function. This last statement is true only if we make an appropriate assumption as to the set of values of x and might well be taken as an example to justify the *Universal Mathematics* definition of function which includes an explicit reference to the domain of the function. The symbol $\int_{-3/2}^{x} 2\,dt$ is distinguishably different from the symbol $2x+3$ but is as trivially different a representation of the function as $(6x+9)/3$, although again it is certainly a different rule.

The system of equations $dy/dx=2$ and $y(0)=3$ is another representation of this same function. If we let t represent any real number and write $x=t$, $y=2t+3$, we obtain a so-called parametric representation of the function $2x+3$. Another slightly more complicated-looking parametric representation is $x=\log t$, $y=2\log t+3$; here, t is a positive number. Thus, perhaps, it is clear that the number of possible distinguishably different representations of even this simple function is infinite. Perhaps it is appropriate to note here that, although $(6x+9)/3$ and $2x+3$ are different functions, if a function is a rule and also if the symbol itself is the function, the functions will be equal. This is true according to the definition that two functions are equal if they are defined for the same values of x, and if for each x for which they are defined their values are the same.

If we define the graph of the function $f(x)$ to be the set of points $(x, f(x))$, then we can represent the function by a picture. For example, with reference to a Cartesian co-ordinate system, the graph of $2x+3$ is a straight line with slope 2 and y-intercept 3, and the picture on this page represents the function. This picture is a representation of the function in a quite different sense, however, than that in which the symbol $2x+3$ represents the function. The symbol $2x+3$ is a precise representation,



and the picture by itself is at best a crude approximation.

Tables are familiar representations of functions, e.g., tables of squares, square roots, cubes and cube roots, logarithms, and trigonometric functions. For the function $2x+3$ we exhibit a miniature table (and we could, of course, construct a big one).

x	-3	-2	-1	0	1	2
$2x+3$	-3	-1	1	3	5	7

This, like the picture, is a crude approximate representation of the function $2x+3$, but it is approximate in a different sense from that in which the picture is approximate. The function $2x+3$ has a value for every real number x, and the table only shows values at a few discrete points. There actually are infinitely many functions radically different from the function $2x+3$ for which this table would be correct. Tables of values of the trigonometric and logarithm functions are approximate both in this sense and also in the same sense as the picture, since with relatively few exceptions each tabular value is an approximation.

The symbol $7x-1$ is also distinguishably different from $2x+3$. This symbol represents a truly different function; yet these two functions are closely related in what is to us a perfectly obvious way. They are both members of a class or set of functions which we call linear functions. This class exists by virtue of the definition: a function is a linear function if and only if it is representable in the form $ax+b$ where a and b are any numbers. We

take the trouble to create this class because these functions have certain properties that not all functions do have. The most familiar of these properties is that the graph of any linear function with reference to a Cartesian co-ordinate system is a straight line. This property is possessed by no other functions and is for this reason often said to characterize the class.

If we regard $2x+3$ as a variable, its range, i.e., its set of values, is the same as that of x itself. Thus it can play the role of the variable x in the function $7x-1$. This leads to the symbol $7(2x+3)-1$ which also represents a function, which it is convenient to call a "function of a function" or a "composite function." Now $7(2x+3)-1=14x+20$, so that this composite function is also a linear function. This illustrates another property of this class of functions, viz., a linear function of a linear function is a linear function. This property does not uniquely characterize the class because it is property possessed also by many other classes of functions.

This function, $2x+3$, is also a member of other classes or categories of functions. For example, consider the class of all functions which are representable in the form $a_0x^n+a_1x^{n-1}+\cdots+a_n$ where n is a non-negative integer and $a_0, a_1, \cdots, a_n$ are numbers. This class we call the polynomial functions. How does one decide for any given function whether it does or does not belong to this class? Although the definition is very specific, if one recalls that any function may be representable in an infinite variety of ways, he may be led to wonder if it might not at least occasionally be difficult to reach a decision for a given function.

Such is indeed the case. However, for the symbol $2x+3$ it is, we would say, perfectly obvious that the function does belong to the class of polynomial functions. Among the properties of this class of functions there is one that high school algebra students readily recognize. Interpreting $2x+3$ and $7x-1$ as numbers, we can form the sum, the difference, and the product of these numbers. We know that if we do this the result can be represented in the form $a_0x^n+a_1x^{n-1}+\cdots+a_n$ with n a non-negative integer and $a_0, a_1, \cdots, a_n$ some particular numbers. That is, the sum, difference, or product of any two polynomial functions is a polynomial function. Also, as in the case of linear functions, a polynomial function of a polynomial function is a polynomial function.

What is an algebraic function? A precise definition can be given very simply in terms of polynomial functions. If $P_0(x), P_1(x), \cdots, P_n(x)$ are polynomial functions, if n is a positive integer, if $P_0(x)$ is not identically zero, and if y is a function such that

$$P_0(x)y^n+P_1(x)y^{n-1}+\cdots+P_n(x)=0$$

for all numbers x, then, and only then, y is an algebraic function of x. A less sophisticated statement, but one generally more useful for the high school student, is that if a function y can be represented by means of a formula involving only a finite number of algebraic operations (i.e., additions, subtractions, multiplications, divisions, and extractions of roots) on x and other numbers, then y is an algebraic function of x. It is a consequence of a property of polynomial equations that not every algebraic function (according to the definition above) can be represented by such a formula.

It is by no means obvious and not at all trivial that the trigonometric functions, the logarithm function and the exponential function are not algebraic functions. These functions, together with the algebraic functions and the composite functions which can be formed from all of these, are the so-called elementary functions.

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1. The National Council of Teachers of Mathematics, *Insights into Modern Mathematics*, Twenty-third Yearbook. Washington, D.C., 1957, p. iii.
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4. Artin, E., *A Freshman Honors Course in Calculus and Analytic Geometry Taught at Princeton University.* Committee on Undergraduate Program, Mathematical Association of America, 1957.
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Letter to the editor

Dear Editor:

On page 388 of The Mathematics Teacher for May 1959, Professor Hoyt states that he would be interested in seeing other proofs of the theorem that he has there proved. I offer two solutions.

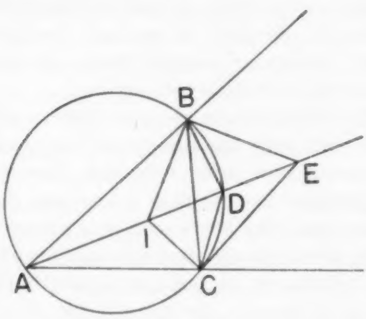

Figure 1

Let I be the incentre, E the e-centre opposite A, of the triangle ABC.

Let the straight line which bisects A internally, viz., AIE, meet the circumcircle of ABC in D.

It is required to prove that D is the midpoint of IE.

Proof I:

CI, CE, being the interior and exterior bisectors of C, are at right angles.

Similarly, BI, BE are at right angles.

Therefore, the circle on IE as diameter passes through B and C.

$$\angle BEC = \text{supplement of } \angle BIC$$
$$= \angle IBC + \angle ICB$$
$$= \frac{\angle B + \angle C}{2}$$
$$\angle BDC = \text{supplement of } A$$
$$= \angle B + \angle C.$$

Therefore, D is the centre of the circle $BIEC$ and so the midpoint of IE.

Proof II:

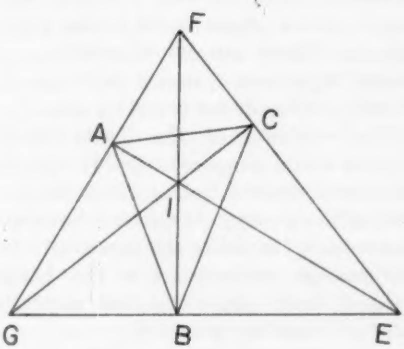

Figure 2

Let E, F, G be the e-centres opposite A, B, C respectively.

Then, EA, FB, GC are the altitudes of the triangle EFG, and I is its orthocentre.

Therefore, the circumcircle of triangle ABC is the nine-point circle of the triangle EFG.

Therefore, the midpoint of IE lies on this circle.

Not only does the midpoint of IE lie on this circle but the midpoint of EF does, too.

We may now combine these results in the following theorem.

The midpoints of the six line segments formed by joining in pairs the four points, the incentre and the three e-centres of a triangle, lie on the circumcircle of the triangle.

Yours faithfully,

N. L. Maseyk
Hutt Valley High School
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An experimental program in mathematics

COY C. PRUITT, *Tulsa Public Schools, Oklahoma.*
A description of what one school system is doing with the more able students, with some evaluation by follow-up.

I THINK a more appropriate title for this paper would be "A Program in Mathematics for the Academically Gifted Student."

First, let us consider something of the background of this program. Tulsa is a relatively young city with a school population of more than 64,000 in the public schools. There are 62 elementary, 15 junior high, and 7 senior high schools. Tulsa operates under the 6-3-3 plan.

The students of the Tulsa Public Schools would compare favorably with the average in cities of similar size in this section of the country. We do not have special schools for either the superior or below-average students. For the handicapped, both physically and mentally, special classes are provided.

As in all cities, schools in different sections vary as to economic, social and academic background of the school population. You will find as much difference among the schools within the city as that between widely separated cities.

Our program is one that any average school with an enrollment of 200 or more per grade could put into operation. I do not think this type of program would prove economically satisfactory in a small school.

In the fall of 1954, Edison Junior-Senior High School opened its doors for the first time. While the junior and senior sections are housed in the same building, they occupy separate wings and have distinct organizations. The combined enrollment is approximately 3,700 students. Dr. Hiram Alexander serves as principal for both schools.

During the fall semester of 1954–55 a student transferred to Edison from another school. He was in the eighth grade, and an excellent student. His parents reported to the advisory staff at Edison that their son seemed to be just "marking time" in his mathematics class, and asked if he could try first-year algebra.

This student was given an algebra aptitude test with the understanding that if his scores were sufficiently high, he would be permitted to enroll in a first-year algebra course. The boy's score was extremely high, so he was enrolled in beginning algebra. Although he entered a month late, he soon became the top student in the class.

Here was conceived the idea that perhaps there were many students now enrolled in the seventh grade who were capable of going directly into algebra in the eighth grade. It was decided to try out this program beginning in the fall of 1955. A teacher, Mrs. Dorothy Salter, was chosen who believed wholeheartedly in such a program, and was enthusiastic about it, and the machinery was set in motion for initiating it.

Two tests were given to all seventh-grade students near the end of the school year, the Stanford Achievement Test and the California Algebra Aptitude Test. All students ranking at the ninth-grade level, or above, on the Arithmetic Reasoning section of the Stanford Achievement Test

and above the ninetieth percentile on the California Algebra Aptitude Test were considered eligible for algebra.

This group was called together by the principal and the program was explained. Some additional prerequisites were also set up. They were as follows:

(1) The student must be interested in mathematics with a planned major in this subject.
(2) He must have the written approval of his parents.
(3) He must have a recommendation from his seventh-grade teacher. This recommendation was based upon work in seventh grade, work habits, I.Q., and enthusiasm for work.

Sixty students met the conditions listed above. As a result, two classes in eighth-grade beginning algebra were started in the fall of 1955. The average grade level of this group on the Stanford Achievement Test in Arithmetic Reasoning was 10.7. On the California Algebra Aptitude Test the average percentile was 93.3. The average I.Q. was 119.0.

The first three weeks' work consisted of an intensive study of the work normally taught in the eighth grade and not taught in previous or subsequent courses. The remaining part of the year was the traditional work normally taught in first-year algebra.

The Stanford Achievement Test, the Lankton First Year Algebra Test, and the Co-operative First Year Algebra Test were given to these two classes in the spring of 1956. (The algebra tests were also given to the regular ninth-grade algebra students.) The average score on the Stanford Test, Arithmetic Reasoning Section, was 12.0. The average percentile rank of the students on the Lankton Test was 92.3, and on the Co-operative Test 95.5. This was considerably higher than the scores made by the regular ninth-grade algebra students.

In the ninth grade these experimental classes were given the regular second course in algebra. Of the original 60, 53 were left, seven having moved from the city. Content of this course consisted of review of basic algebra; linear equations in one, two and three unknowns; factoring; fractional and negative exponents; quadratic equations; determinants; logarithms; imaginary and complex numbers; binomial theorem; progressions and series, powers and roots; mathematical inductions; variations; theory of equations; graphs of functions in one, two, and three unknowns; and probability.

The Co-operative Advanced Algebra Test was given to this group in the spring of 1957, near the completion of second-year algebra. The average percentile rank of the group was 92.7.

In the school year 1957–58 this group took combined plane and solid geometry. There were now 46 in the group. Three dropped back to regular work; three did not enroll at Edison; and one withdrew from school. However, approximately 77 per cent of the original group finished the work in 1957–58.

Unfortunately, there were no textbooks available in which plane and solid geometry were integrated into a one-year course. The teacher in these two sections spent about three-fifths of the year on plane geometry and two-fifths on solid. Both the Co-operative plane and solid geometry tests were given to this group in the spring of 1958. The average percentile rank of the group on the plane geometry test was 93.6, and on the solid 94.8.

As juniors in senior high school this group took trigonometry and college algebra, using a textbook in which the two subjects were more or less integrated. The average percentile rank of the group on the Co-operative Trigonometry Test was 79.6.

These students are now seniors and taking mathematical analysis with some attention to probability and statistical inference. Of the original 60 who started in the eighth grade 60 per cent remain.

In the fall of 1956 four classes in eighth-

grade algebra were started, and there have been four new classes added each year since then. The method of selecting the students was similar to that in selecting the first group. However, this year the Iowa Algebra Aptitude Test was given instead of the California Aptitude Test. Also, all parents were called in and the program was explained to them individually by the school counselors.

As a result of the success at Edison, six other junior high schools have organized classes in first-year algebra at the eighth-grade level. This new program seems to be working out quite satisfactorily. Everyone connected with it seems enthusiastic and well pleased with the results.

According to the present plans, a follow-up will be made of the college work of the members of the first three groups. If any do not go to college, they will be followed in their chosen vocation. At present 100 per cent indicate that they are going to college.

In order that a better evaluation of this program can be made, Dr. Hugh Livingstone, Director of Research in the Tulsa Public Schools, has selected 60 students who started in the same class with the accelerated group but did not take eighth-grade algebra. As far as possible these 60 students were selected so that their grades and abilities rankings on tests paralleled those of the algebra group. It is the intent to compare these two groups as to selection of courses, achievement in high school, and selection and achievement of work in their first two years in college.

The per cent of the original group that remains is rather high, considering the economic trends of the past two years. It is felt that if 50 per cent of the original group remains, it will provide sufficient information for a more detailed analysis of the value of the program.

One inportant aspect of this first group is that on all tests given they have consistently outscored those students in the regularly enrolled classes. Last year on the Quantitative Section of the Iowa Educational Development Test the average of this group was the 93rd percentile.

This program was started before the full impact of the pressure for modernizing mathematics had begun. Hence, much of the content of the various courses has followed more or less the traditional lines as to subject matter. However, there is no valid reason why this set-up would not be ideal for introducing some of the newer concepts.

Dr. Buswell of the University of California in a report given at the meeting of the National Council of Teachers of Mathematics in Cleveland made the statement that many of our elementary children could learn all the arithmetic necessary by the end of the sixth grade. It seems that we have quite a large number of students who mark time in the traditional seventh- and eighth-grade work. There is no good reason why the essential parts of seventh- and eighth-grade work cannot be combined into one year for those students who like mathematics and indicate that they have a fair degree of success in the subject.

This would mean that this group could gain one additional year of mathematics by the time they finish high school. Many colleges are strongly recommending that high school graduates who intend to major in engineering or some fields of science should have an introduction to calculus and analytics. Beginning algebra in the eighth grade is one way to provide this experience.

Another approach to this increased demand for better prepared students in mathematics is to combine plane and solid geometry into a one-year integrated course. This is being done in two schools in Tulsa. The courses are on an experimental basis at present. By the end of the 1959–60 school year we hope to have some tangible evidence concerning materials for this course. Combining these two subjects will allow one semester for some form of advanced mathematics in the twelfth grade for our regular classes.

What are some of the advantages for

beginning algebra in the eighth grade for superior students?

(1) These students are enthusiastic about mathematics; they are eager to learn and have the necessary "push."

(2) They are capable. All our tests given to date indicate that they do better work than the regular students who are a year older.

(3) They will receive an additional year's work in mathematics. This should provide adequate background for almost any college course.

(4) These students will be able to do better work in the physical sciences because of their mathematical background.

What are some of the problems?

(1) Some of the students are not sufficiently matured and become emotionally disturbed at trivial upsets.

(2) A tendency of some students to feel that they belong to a superior group. Some snobbishness may develop.

(3) Teachers have a tendency to expect too much from the group in later courses.

(4) A tendency on the part of some parents to pressure their children beyond their capabilities. In some instances emotional disturbances may be caused by such pressures.

It is not known what effects the results from the Maryland Study, the Illinois Study, the School Mathematics Study Group, or similar studies may have on this program. To date the program seems to be working out satisfactorily and solves an old problem in the mathematics curriculum—namely, what to do in the eighth-grade mathematics classes.

Have you read?

Cliffe, Marian C. "Mathematics Evaluation in a Large City," *The Bulletin of the National Association of Secondary School Principals*, May, 1959, pp. 161–165.

It has been said that those things we cannot measure we cannot know. Certainly without an evaluation of our mathematics program we are not fulfilling our obligation. This article presents a good description of the how and the why of the mathematics evaluation program in the Los Angeles school system.

The evaluation program tries to find the answers to these questions: Does the student grow in competence, accuracy, interest, appreciation, vocabulary, and problem-solving insight? Does he do mathematics commensurate with his ability? The evaluation program also measures curriculum, enrichment, modernization, teacher competence, breadth of offering, and maximum mathematical experiences. All this is done through a series of measurement techniques including achievement tests, teacher-made tests, tests made by committees of teachers, and tests made especially for the schools by authors of the textbooks in use.

You will want to see how such techniques might fit an evaluation program in your school. This issue of the *Bulletin* is devoted to mathematics.—*Philip Peak, Indiana University, Bloomington, Indiana.*

Neeley, J. H., "What to Do About a New Kind of Freshman," *American Mathematical Monthly*, Aug.–Sept., 1959, pp. 584–586.

After considering the discussion of whether or not calculus should be taught in the high school, one needs to arrive at his own answer for his own school. This article presents some of the problems encountered by Prof. Neeley. He concludes that the students who have studied calculus in high school are bright, but the calculus they know reckons only with "how," not "why." This puts the student at a disadvantage, for nowhere can he get the foundation and not meet duplication. Advanced standing produces the same problem because of cutting across instructional areas. Course failures increase when students who present college credit from high school are advanced.

Prof. Neeley proposes a special type of college for some and a limiting of mathematics in high school to that traditionally considered as appropriate. Be sure to read this, but draw your own conclusions.—*Philip Peak, Indiana University, Bloomington, Indiana.*

The trapezoid–and area

WILFRED H. HINKEL, *East View Junior High School, White Plains, New York.*

An informal approach to the formulas for the areas of the usual polygons studied in the junior high school.

THE USUAL presentation of units in the study of area in eighth-grade work includes the basic concept of square measure, with reference to rectangles and squares; the offering of parallelogram and rhombus; the discovery that the triangle may be thought of as half of one of these quadrilaterals; the thought that the circle may be considered (but hastily, lest we get into limits) as the sum of a great number of triangles; and then the introduction of the trapezoid.

Area formulas can be developed nicely from the basic concept, from square to triangle easily, to circle less easily; but too often the formula for trapezoidal area is merely required to be memorized, without relating it to its predecessors. Sometimes a trapezoid is transformed by producing its twin, but inverted; the resulting parallelogram is then examined for *its* area; from this the formula for the area of a trapezoid can be evolved.

With a bright eighth grade, one which is interested in concepts and in relationships ("Last year, our teacher said a square is a kind of rectangle. Is it?"), the whole problem of area can be covered in a different way. The development turns the usual approach upside down, relates the trapezoid to the other figures, and can generate much interest.

A challenge to calculate, arithmetically, the easily-found sum of, say, the first seven natural numbers is offered and quickly met.

$$1+2+3+4+5+6+7=28$$

A challenge, now, to develop from this solution an algebraic formula to fit this problem and its solution, one which involves relationships between the numbers named, their positions in the series, and their quantities, may offer obstacles. With prodding, the formula for the sum of a series may evolve:

$$S=\tfrac{1}{2}n(a+l)$$

where

n = the number of terms in the series,
a = the initial number of the series, and
l = the final number.

After trial of the formula on easily-tested problems which seek, for example, the sum of the first ten numbers, or the first fifteen or twenty, acceptance of the validity of the formula for such a series of any length is not difficult.

If the question now be asked whether the formula holds for a portion of the series of natural numbers which begins elsewhere than at 1, say at 10, and ends with 20, the matter of proof by arithmetic means is simple.

Now if we file this formula and produce a diagram of the end of a stack of cylindrical tiles (*Fig.* 1), it is not difficult to con-

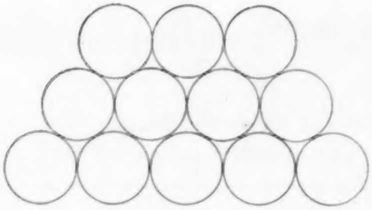


Figure 1

vince that the same formula applies in the problem of quickly counting these tiles.

Let us now enclose our stack of tiles with a minimum number of straight lines (the edges of the crate in which they were shipped?)—see *Figure* 2.

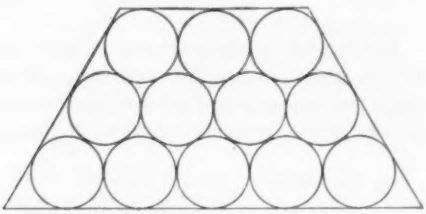


Figure 2

If we now equate the number of tiles in the bottom row of our stack with the units of length of the lower base of our box; the number of tiles in the top row of the stack with the units of length of the upper base of our box; and the number of rows of tiles with the units of length in the height of our box—we have only to substitute the standard symbols used in the usual trapezoid area formula for the symbols used in our formula for the sum of a series, thus: where

$$\left\{\begin{matrix} n=h \\ a=B \\ l=b \end{matrix}\right\} : \left\{ S=\tfrac{1}{2}n(a+l) \right\}$$

$$= \left\{ A=\tfrac{1}{2}h(B+b) \right\}.$$

This relationship having been arrived at, it is now possible to view other plane figures to discover if there is a kinship.

In the rectangle:

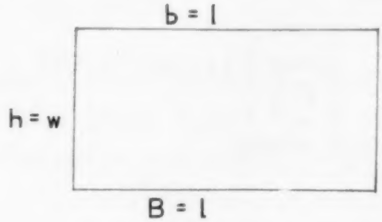


Figure 3

where

$$\left\{\begin{matrix} h=w \\ B=l \\ b=l \end{matrix}\right\} : \left\{ A=\tfrac{1}{2}w(l+l)=lw \right\}.$$

In the square:

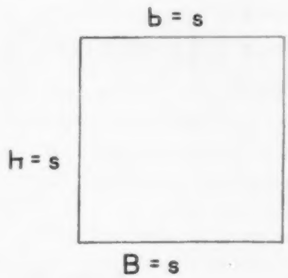


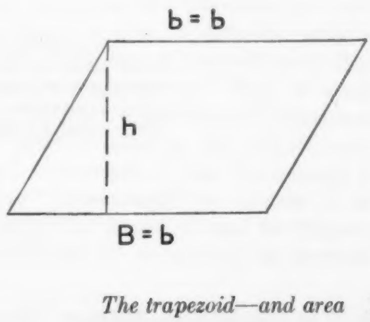
Figure 4

where

$$\left\{\begin{matrix} h=s \\ B=s \\ b=s \end{matrix}\right\} : \left\{ A=\tfrac{1}{2}s(s+s)=s^2 \right\}.$$

In the parallelogram and rhombus:

Figure 5



where

$$\begin{Bmatrix} h=h \\ B=b \\ b=b \end{Bmatrix} : \{A=\tfrac{1}{2}h(b+b)=bh\}\cdot$$

In the triangle:

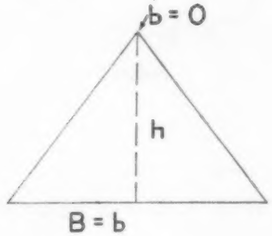


Figure 6

where

$$\begin{Bmatrix} h=h \\ B=b \\ b=0 \end{Bmatrix} : \left\{A=\tfrac{1}{2}h(0+b)=\frac{bh}{2}\right\}\cdot$$

In the kite:

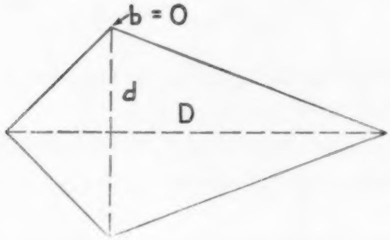


Figure 7

where

$$\begin{Bmatrix} h=\tfrac{1}{2}d \\ B=D \\ b=0 \end{Bmatrix} : \left\{A=(2)(\tfrac{1}{2})(\tfrac{1}{2})d(D+0)=\frac{dD}{2}\right\}\cdot$$

And, about this moment, out of the back of the room comes the question: "Is a circle really the sum of a lot of degenerate [a new word, recently] trapezoids?"

From *The Mathematics Teacher* of thirty years ago

"... The fact remains that a large number of pupils cannot handle fractions either common or decimal, and that lack makes their algebra work very difficult. There is evidence also of weakness in the very fundamentals of arithmetic. ...—*Joseph B. Orleans and Jacob S. Orleans,* "A Study of Prognosis in High School Algebra."

"No mathematician can be really sophisticated until he knows that Weierstrass (1815–1897) gave an example of a continuous function which is nowhere differentiable."—*R. P. Agnew,* Differential Equations.

Individualized algebra

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How can students learn to capacity without ability grouping?

IS IT possible to develop the abilities of each student to the maximum by individualizing the instruction in heterogeneously grouped classes?

It was in an attempt to individualize instruction that we began our work. It was our feeling that the goals of all mathematics teachers should provide for the following:

1. Each student should progress as fast and as far as possible.
2. The program should contain maximum enrichment.
3. Each student should enjoy learning mathematics.
4. Each student should have a successful mathematics experience.
5. The mathematics class should be an experience in democratic living.
6. Growth should take place in many areas, i.e., reading, manners, vocational interest.
7. Individual discovery of mathematical concepts should be developed.

Three factors which influenced our original thinking should be mentioned. One was the progress made by New York City students as a result of the developmental mathematics program. The success of individualized reading programs was another. The third factor was the apparently widespread hatred and fear of mathematics which we have observed in college and high school students. We have no documentation but feel that this view is shared by many educators. Because we started the school year with a specific point of view, and since we made no attempt at matching the students, this project is not offered as a scientific experiment. All we have to say is, "This is what we did, and this is how it turned out."

Our study was carried out in two suburban high schools near New York City. Each of us had two first-year algebra classes, one in which we tried our best to emphasize the individual development of the students, and the other in which we employed group teaching techniques. The latter was the teacher-presentation, student-recitation type of instruction with the class in a single group. We each taught one class of each type. The students were not selected according to any criteria; they were ninth- and tenth-graders who could have elected general mathematics rather than algebra, if they so desired. Many of the tenth-graders were repeating algebra, having failed the subject in grade nine. The I.Q.'s ranged from 80 to 126, but since some of the students had not been tested, and since the scores we did have were from a number of different tests, all we can say is that it seemed to us that, except for Experimental Class *B* which seemed to have a very high proportion of very bright students, our pupils were typical of average classes. We decided to try our "experiment" on classes of 20 (Experimental Class *A*) and 26 (Experimental Class *B*) which met at approxi-

mately the same time of day. The other classes, one of 24 and one of 25, were designated controls *A* and *B*.

In planning the study, we decided that everything we would do would be designed to give the student the primary responsibility for learning, while the teacher maintained responsibility for leadership. We began by getting to know our students as well as we could. In the first weeks of the year each class worked as a single group and concentrated upon fundamentals. Then things began to change. The students moved into small groups formed on a variety of bases. Some students found a common vocational goal the binding force for a group. Others were simply friends who wanted to work together. Still others were interested in the same areas of study. We did our best to maintain these developed groups in as fluid a state as possible, encouraging student change at every opportunity. In their groups the students selected topics from the New York State syllabus. No particular sequence was adhered to, and no two students necessarily followed the same sequence. The student had freedom of choice, within the stated limits, to select his own area of work. It was also possible for related topics to be selected outside the required syllabus with the approval of the teacher.

The students asked about homework. We put the question back to them. If they thought it was necessary, they should make their own assignments and keep careful records of all that they did. If they did not deem it necessary, there was no assignment of homework. The student's primary goal was to learn as much algebra as he could. If homework would help achieve the goal, the students should undertake work at home. It was a decision they had to make for themselves.

They asked about tests. Discussion of the question brought about the conclusion that when a student felt that he had achieved mastery of a topic, he would request to be tested. The groups could ask for a test for all or any of its members.

The students wondered about topic selection. Planning sessions were held on both a long- and short-term basis; a form was developed to help the students keep track of their planning, and to make comparisons with the actual amount of work they had completed.

What effect did this different approach have upon our students? We found that most students can function well in a democratic environment, sensing that they have an opportunity for increased accomplishment rather than a chance to "goof off." De-emphasis of grades, lack of necessity to keep up with the next fellow, and elimination of external punitive measures generally made for a better learning situation, not a poorer one. We feel that it enabled us to accomplish the following:

1. self-discovery of concepts
2. improvement of reading through work in problem solving
3. improvement of social relationships through co-operative group work
4. improvement of manners through unobtrusive individual instruction
5. development of self-discipline since the major punitive measure was the inevitable result of one's own behavior: a lack of sufficient completed work and failure to advance towards one's stated goal.

How did we make out with our "experiment"? In order to have an objective measure of scholastic achievement, we administered the Co-operative Algebra Test (Educational Testing Service, Princeton, New Jersey). The scores of Form *Y*, given in January, and Form *T*, given in June, are presented in the accompanying table. As can be seen, despite the problem of adjustment to a new approach, the experimental groups held their own. The differences were not statistically significant. While we do not believe any definite conclusions can be drawn from the data, it does seem obvious that emphasizing a democratic and individual approach did not retard student achievement.

Summary of scaled scores on the Co-operative Elementary Algebra Test*

Group	*Form*	*No.*	*Range*	*Median*	*Mean*	*St. Deviation*
Exp. A	Y	20	35–59	46.0	46.5	6.24
	T	20	37–69	51.0	52.8	6.55
Exp. B	Y	26	35–71	49.5	51.8	8.90
	T	26	31–76	54.5	54.9	11.04
Con. A	Y	24	31–65	44.5	45.1	8.80
	T	24	42–71	53.5	54.7	8.63
Con. B	Y	25	35–67	49.0	47.8	7.93
	T	25	34–70	50.0	52.0	8.83
Total Exp.	Y	46	35–71	49.0	48.9	8.12
	T	46	31–76	52.5	53.9	9.53
Total Con.	Y	49	31–67	46.0	46.3	8.60
	T	49	34–71	52.0	53.3	8.36

* Educational Testing Service, Princeton, N. J.

We feel that achievement in the "experimental" groups could be increased by the development of effective discovery materials. Increased experience with this approach will also result in more and better learning experiences. The discovery exercises in our textbook were not as effective as we had expected. As the students progressed and the material began to increase in difficulty, self-discovery of concepts became correspondingly difficult. One task we have set for ourselves is the creation of adequate materials to develop student discovery. If the use of these materials results in greater achievement by our experimental groups, we will have evidence of their need.

There were many results worthy of note in areas other than achievement in subject matter. The grouping on a random basis encouraged a sharing of learning by all group members. In the beginning there was much movement in and out of groups, but later the group structure became more static. Even in static groups, there was evidence of a differentiation of goals within the group. Thus, although there were students who preferred to work together, they did not work on the same subject matter.

One of the more startling results to us was that the sequential order of the text was not followed to any noticeable disadvantage. This leads us to suspect that the sequential order in which teachers feel they must develop the subject is not sacred—an hypothesis to be tested further. If our later studies support this finding, teachers should be free to maximize individual development and have an opportunity for creation of the highest possible interest level on the part of each student.

Pupil planning is in need of a more careful introduction than we were able to make in our study. After an initial burst of careful planning on the part of most students, there was a lack of continuous planning. It seems that inexperience in this area led to quick discouragement on the part of many students. The use of a form sheet to aid planning for the student did result, however, in several cases of intelligent preparation. It is clear that if pupils are to plan their semester's work skillfully, they will first have to undergo an intensive orientation period. This is another item for further study.

The self-assignment of homework went well. Most students clearly saw their responsibilities and carried them out in good

order. The problem of test readiness also met with general success. In one group the time lapse between examinations came under consideration. After much discussion, the group settled this problem by proposing a two week limit: if any student did not request a test over a two-week period, the teacher was instructed to give him a test on the material he had covered thus far. In regard to tests, it is interesting to note that there was virtually no evidence of cheating in the experimental groups. In our control groups, we had the usual task of careful proctoring; not so in the experimental. This behavior was also borne out in the homework assignments. Copying, occasionally observable in the control group, was, so far as we can tell, nonexistent in the experimental classes.

Conclusions

As a result of our study we definitely feel that individualization of instruction in heterogeneously grouped classes is possible. The advantages of such instruction were:

1. The bright student could move ahead, while the slow students were not discouraged since they were under no pressure to maintain another student's pace.
2. As teachers, we were more aware of the progress of our students than we have ever been before.
3. The students genuinely appreciated:
   a. sharing their learning experiences
   b. individual tests at their request
   c. self-selection of topics
   d. working at a rate commensurate with their ability.

   All of these helped to dispel fear and overcome a distaste for mathematics in general.
4. A greater sense of responsibility was apparent. As far as we can tell, this was a result of the democratic procedures and the general atmosphere.
5. Self-motivation was the stimulus for learning, rather than the usual extrinsic motivation of grades, etc.

On the other side of the ledger we note:

1. In order to carry out this study we found that we had to do a tremendous amount of work. Planning, testing, giving help when requested—all involved large quantities of time and energy.
2. We had too little discovery material and what we did have was inadequate. Effective materials are sorely needed and will be developed in the near future.
3. A period of adjustment was necessary for the student to make the proper transition from the patterns developed in previous classes to the approach of the experimental group.
4. Generally speaking, the group as a whole didn't cover material as fast as we had expected. We feel the lack of effective materials was the main cause.

Summary

We carried out this study to determine whether or not it would be possible to individualize instruction to the point where students select their own topics for study, decide upon their test readiness, assign their own homework, and in general are responsible for learning. We also tried to determine the possibility of employing a democratic approach. With all this emphasis upon the development of maturity, our students did not allow their study of mathematics to become secondary. They learned at least as well as their contemporaries in the control group. We have made a start and are going to try to improve upon it.

A survey of teachers' opinions of a revised mathematics curriculum

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and ROBERT C. FISHER, *Ohio State University, Columbus, Ohio.*

Do high school and college teachers differ on whether the high school curriculum should be revised?

ON MAY 14, 1959, the Third Annual Symposium on Engineering Mathematics was held at Ohio State University under the sponsorship of the College of Engineering and the Department of Mathematics. The Symposia have as their broad objective the coordination and improvement of mathematics teaching, particularly for students in the engineering and physical sciences.

The Third Symposium was attended by approximately 280 persons, primarily high school mathematics teachers and college instructors in mathematics and engineering, from throughout Ohio and its neighboring states. This year the main topic of discussion was the four-year curriculum in mathematics as taught in the high schools. Particular emphasis was given to an evaluation of the suggestions set forth in the Final Report of the Commission on Mathematics of the College Entrance Examination Board, entitled "Program for College Preparatory Mathematics."

The principal speaker at the morning session was Professor Saunders MacLane of the Department of Mathematics, University of Chicago, who gave his evaluation of the Commission's Report and how to use it. The luncheon session featured Robert E. K. Rourke, Executive Director of the Commission on Mathematics of the College Entrance Board, who explained the background and objectives of the Commission.

Following the luncheon session, the persons attending the Symposium were invited to participate in discussion groups, each of which consisted of approximately a dozen people. They were assigned to discussion groups on an individual basis so as to be sure that each group contained some high school mathematics teachers, college mathematics teachers, and engineering college teachers. A number of favorable comments from the participants in these discussion groups indicated that the heterogeneous grouping was very desirable. Each discussion group had a discussion leader assigned to it. These leaders were chosen to include high school mathematics teachers and supervisors, college mathematics teachers, and engineering college teachers.

The main topic of discussion for these groups was the Commission's Report and Professor MacLane's interpretation of the Report as given in the morning session. In order to have some specific information regarding the feelings of those attending the Symposium toward the Report of the Commission, each participant in the discussion groups was asked to fill out a questionnaire. On the questionnaire the respondent was asked to identify himself by name and position. The questionnaire contained 15 questions. These questions were formulated directly from statements made in the Commission's Report so that the response of an individual might be

taken as an indication of the degree of acceptance of the Report. The following 13 questions asked for a simple "yes" or "no" answer, and invited the respondent to make comments.

1. Do you agree that an extensive revision of the present traditional high school mathematics courses is necessary?

2. Do you agree that every college-capable student should take at least three years of mathematics in high school (regardless of his major interest)?

3. Do you agree that homogeneous grouping of students by ability is highly desirable?

4. Do you agree that calculus is primarily a college subject, and that only schools with exceptional staffs should undertake the teaching of calculus and then only to exceptional students?

5. Do you think it is reasonable and realistic to expect a level of mathematical rigor in the 9th grade algebra that is higher than that now expected in the 9th grade?

6. a. Do you agree that more emphasis should be placed on deductive reasoning in 9th grade algebra?

b. And do you think this increased emphasis can be carried out without incurring a reduction in the manipulative skills of the student?

7. Do you agree that coordinates should be introduced into the 10th grade geometry rather early, say after 6 weeks?

8. Do you feel that the logical deficiencies in the treatment of plane geometry by Euclid as mentioned in the Commission's Report have been a barrier to the student's understanding of the logical development of geometry?

9. Do you agree that solid geometry should be eliminated as a separate high school course and that it should be integrated into the 10th grade course?

10. Do you believe that other geometries—non Euclidean, projective, finite geometries—should be regularly taught in high schools?

11. Do you agree that rudimentary trigonometry of the right triangle should be included in the 9th grade algebra?

12. Do you agree that the 12th grade course should include at least one semester of the material labeled as "elementary functions" in the Commission's Report?

13. Do you favor the advanced placement program offered by some high schools (in which Ohio State participates) which allows students to take college level math courses while in high school and obtain college credit?

The questionnaire also contained the following two items:

14. Check those topics in the following list that you feel should definitely be taught regularly in the college preparatory program in high school.

a. inequalities	g. determinants
b. absolute values	h. group theory
c. algebra of sets	i. field theory
d. limit concepts	j. differential calculus
e. vectors	k. statistics
f. probability	l. integral calculus

15. How many quarter hours of college credit in mathematics do you think a high school teacher should have as a minimum in his training to be prepared to teach the complete program suggested by the Commission?

Of those attending the group discussion, 186 returned completed questionnaires that were subsequently tabulated. Among those returning questionnaires were 35 college mathematics teachers, 19 engineering college teachers, and 132 high school mathematics teachers and supervisors. The bar graph in *Figure* 1 shows the percentage of "yes" answers to the first 13 questions from the high school teachers, from the college teachers, and from the combined group. The percentages given are based upon the total number of answers for each particular question.

The tabulated responses indicate clearly that there is overwhelming support for the major part of the Commission's Report among the teachers attending the Symposium. Furthermore, there is seemingly no significant difference between the high-school-teacher response and the college-teacher response except possibly on questions 1, 2, 7, and 10. The difference in response on question 1 is the first of three indications that the college teachers are perhaps less inclined than the high school teachers are to feel that "extensive" revision of the curriculum is necessary. The "no" response on question 1 often carried the comment that revision was necessary, but not extensive revision. The difference in response to question 7 can be partly attributed to the fact that the high school teacher is more sensitive to the question of timing in the introduction of coordinate geometry. Comments on some "no" responses on question 7 indicated disagreement with the timing rather than the inclusion of coordinate geometry. The factor of timing also entered in the disagreement with the statement of question 11. The

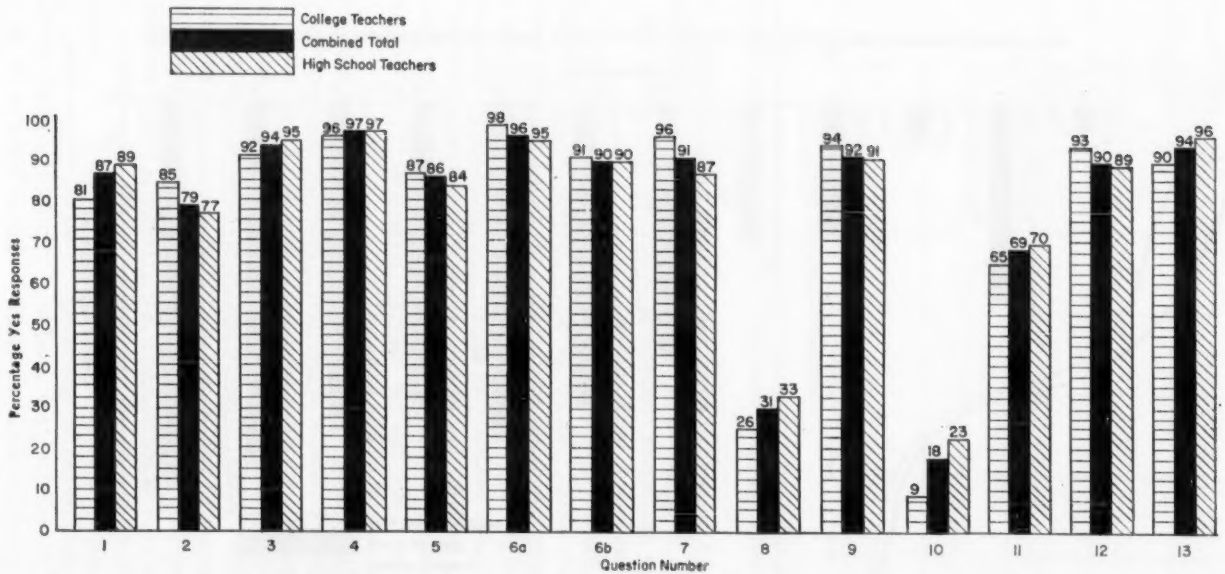


Figure 1. Percentage of "yes" responses to questions 1–13. (Percentage based on total number of responses for each question.)

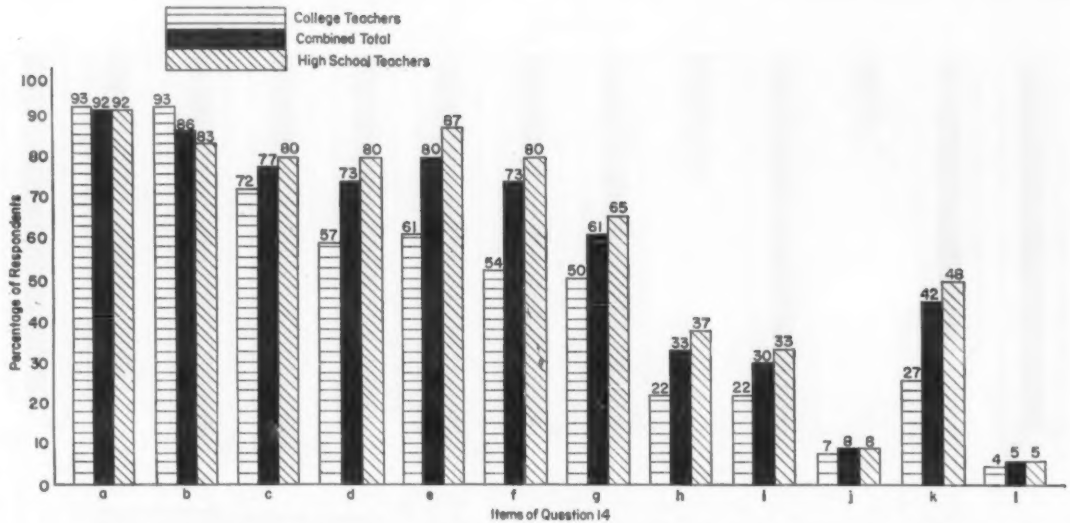


Figure 2. Percentage of respondents favoring inclusion of items in high school mathematics curriculum.

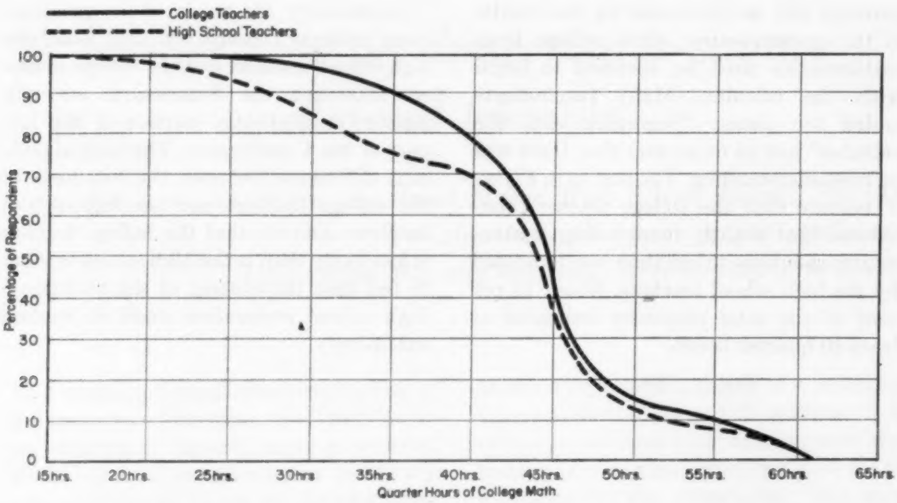


***Figure* 3. Percentage of respondents favoring at least the indicated number of quarter-hours of college-level mathematics training for high school mathematics teachers.**

comments on this question indicated many teachers felt right triangle trigonometry belongs in the 10th grade, not the 9th grade.

The large number of "no" responses on question 8 indicate that neither teacher group seems to agree that the logical defects in Euclid's treatment of geometry can be regarded as a valid reason for modifying high school geometry. Both teacher groups also rejected the suggestion that other kinds of geometry should be taught in high school.

The fact that less than half as many college teachers (percentage-wise) responded with "yes" on question 10 as did high school teachers indicates that the college teachers are less inclined to favor the introduction of nontraditional mathematics into the high school curriculum than are the high school teachers. There is added evidence for this conclusion in the responses to question 14.

In *Figure* 2 a bar graph depicts the percentage of respondents checking the items listed in question 14. It is worth noting that the calculus topics were checked by very few persons. This fact is consistent with the overwhelmingly affirmative answer given to question 4 by the group as a whole. It would thus appear that the Symposium attendants were strongly behind the stand of the Commission that calculus is a college subject. Neither group theory nor field theory was checked by a very large number of people. Except for the first two items in question 14, the topics were checked by a larger percentage of high school teachers than by college teachers. Thus, for example, while 37 per cent of the high school teachers felt group theory should be taught, only 22 per cent of the college teachers agreed. These results again indicate that the college teachers present were less concerned than were the high school teachers about introducing new topics in the high school curriculum.

In *Figure* 3 two curves are drawn depicting the information obtained from question 15. The abscissa of a point on the curve corresponds to a number of quarter hours, and the ordinate of the point indicates the percentage of respondents who indicated at least that number of quarter hours. There was some confusion concerning the meaning of college level mathematics; however, if one agrees with the Commission's Report (and the great majority ap-

parently did, as evidenced by the results of the questionnaire), then college level mathematics must be assumed to begin with the calculus. Many respondents added the phrase "beginning with the calculus" just to make sure that there was no misunderstanding. The curves in *Figure* 3 indicate that the college teachers considered that slightly more college mathematics should be taken than was indicated by the high school teachers. About 75 per cent of the total responses indicated at least 40 quarter hours.

In summary, the results of the questionnaire indicate conclusively that both the high school teachers and the college teachers attending the Symposium strongly support a substantial portion of the Report of the Commission. The only significant differences between the responses of the college teachers and the high school teachers indicate that the college teacher is less likely than is the high school teacher to feel that the content of the traditional high school curriculum need be revised extensively.

From *The Mathematics Teacher* of thirty years ago

"... Gifted pupils should be a constant challenge to their instructor to develop in them all the mathematical insight and skill of which they are capable. They are clearly entitled to an educational opportunity in algebra commensurate with their superior ability."—*L. E. Mensenkamp,* "Ability Classification in Ninth Grade Algebra."

"One who is naive with respect to the question of possible differences is consequences from application of operators in different orders may profit by thinking a little about the operations: (i) insuring an automobile and (ii) driving the automobile into collision with that of a struggling lawyer."—*R. P. Agnew,* Differential Equations.

A further note on dissecting a square into an equilateral triangle

CHESTER W. HAWLEY, *Southwest Miami High School, Miami, Florida.*

The dissection of a square into an equilateral triangle requiring only three cuts, with no part turned over.

As I WORKED through the recent series of papers in THE MATHEMATICS TEACHER on the cutting of figures, making plywood models of the dissections given there,[1] my attention was attracted by the two questions which concluded the solution of Problem IV.[2] Problem IV required us to find a dissection of a square into an equilateral triangle; in the solution presented the square was divided by four cuts into five parts which (after some were turned over) reassembled into an equilateral triangle as desired. The two questions raised (and left unanswered) in connection with this solution were: *Is there a different solution of Problem IV not requiring any part to be turned over? Is there a solution with the square cut into fewer than five parts?*

It seemed to me that one or both of these questions might yield to a little experimentation, so I tried cutting up cardboard squares and assembling the parts in various ways. These efforts were more or less blind, and entirely unsuccessful. Worse, they gave no hint of a fruitful approach to either question. Not hitting upon any clue, even after many trials, I gave up thinking about the questions altogether.

A few weeks later I chanced upon a copy of Steinhaus' *Mathematical Snapshots* (New York: Oxford University Press, 1950). On page 7 of this book of mathematical recreations appeared a sketch of a construction summarized on p. 120 as *Figure* 1. It struck me at once that this four-part dissection of Steinhaus' answered *both* of the questions in the affirmative. The only problem now was to duplicate his construction, for the book gave no hint or direction of any kind on how to draw the cutting lines in the square.

In this note I shall show how, with the help of some experimentation with cardboard models, I came to see the principles behind Steinhaus' construction. Ultimately, I shall give a precise statement of the construction and show that it answers affirmatively both questions cited in the first paragraph. (It turns out, unsurprisingly, that the principles of square dissection put forward in the October 1956 paper mentioned earlier are all that are needed for this construction and verification.)

I do not regard these last matters as most important, however. What concerns me more here is to show *how* the principles of Steinhaus' construction came to me through experimentation. I have found that this mode of discovery is extraordinarily interesting and valuable to geometry students—they grasp quickly what is wanted, they participate in the search and identify possible clues spontaneously, and their success stimulates them "to go into the mathematics of the thing,"—i.e., to verify that their construction is exactly right. What follows, therefore, is primarily pedagogical, and only secondarily mathematical.

[1] See my note in THE MATHEMATICS TEACHER, LI, February 1958, 120.

[2] See "More on the Cutting of Squares," THE MATHEMATICS TEACHER, XLIX, October 1956, 449.

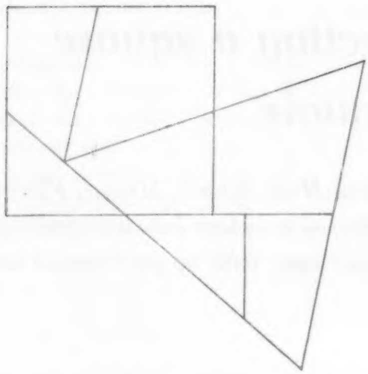


Figure 1

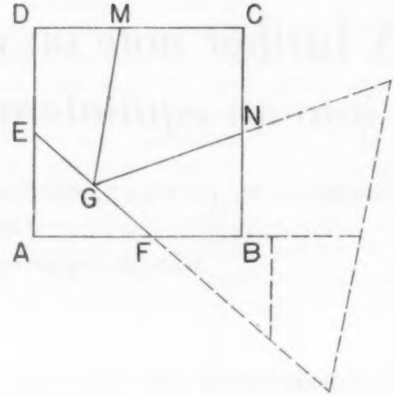


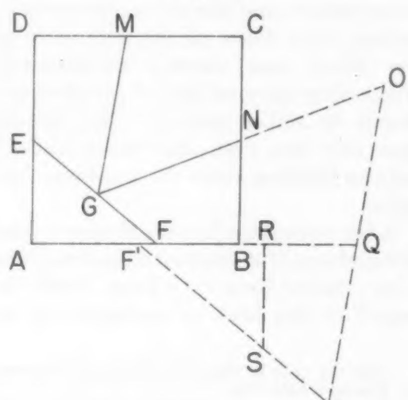
Figure 2

Look at *Figure* 1. Evidently we have here a dissection of the square into *four* parts which *without turning over* reassemble into an equilateral triangle. Hence both of the questions stated in the first paragraph above seem to be answered affirmatively. Our problem is: How to draw the cutting lines in the Steinhaus square of *Figure* 1? To make this problem precise, let us label the relevant points of the Steinhaus square as shown in *Figure* 2. Our problem then is: How to draw segments EF, MG, and NG in the square $ABCD$?

How to begin? A little reflection showed that points E and N were probably the mid-points of side AD and BC. Reassembling "mentally" the four parts of the square into their new positions in the triangle showed at once that this must be the case, since point C must fall on point B (hence making BN and NC congruent) and point A must fall on point D (hence making AE congruent to ED). A square was laid out using this fact. As line EF seemed to offer no clue as yet, it was drawn by "guess." Looking now at line NG meeting EF, an angle of 60° was suggested, since the quadrilateral $FBNG$ was to be a part of the proposed equilateral triangle. The obvious steps of making angles NGM and MGE with 60° were then taken, and the square cut on these three lines. The result is shown in the dotted lines of *Figure* 3.

This was encouraging, and another square was immediately laid out and cut, with one difference—point F was located farther to the left. This time, on reassembling, the "triangle" looked like the dotted lines of *Figure* 4. Putting these two attempts together, we could make some reasonable suppositions: (1) that the square would reassemble correctly into an equilateral triangle if the point F were correctly chosen; (2) that since $(GN) = (NO)$ and $(OQ) = (QP)$, all four of these

Figure 3



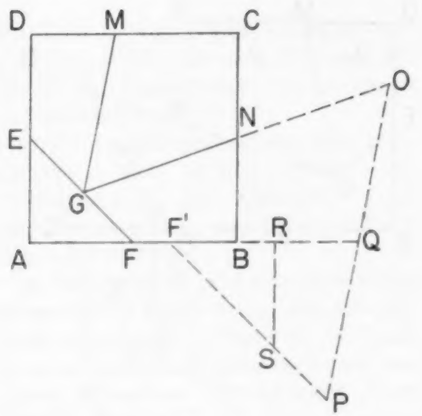


Figure 4

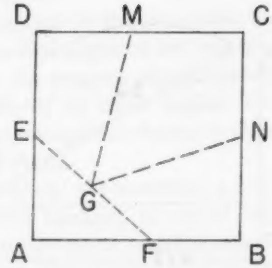


Figure 5

segments might be congruent in the correct construction. If this last were so, we would be well on our way.

Three or four more trials brought the points F and F' coincident to within about 1/64″ (I was using 8″ squares at this point). All of the sides of the triangle seemed straight, measuring $15\frac{1}{8}'' \pm 1/32$, and everything seemed to "fit" well. Furthermore, we had increasing evidence that N and Q were the midpoints of sides GO and OP respectively, and also that the segments they made with the sides were *all* equal. Since our square was much larger than that of Steinhaus, this was encouraging.

When the line FS (which is also EF) proved to be the same length as GN and the other three segments around the triangle, success seemed near at hand, because a little reflection showed that FS could easily be shown to have a length one-half that of GP. (For since $(EG)=(SP)$, then $(GF)+(SP)=(EF)=(FS)$, or $(FS) = (GP)/2$.)

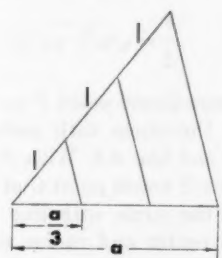
Assuming all this to be true, the problem would be solved if we could construct a segment of length $s/2$, s being the length of a side of the desired equilateral triangle. Using the methods of the original Problem IV in constructing the mean proportional between two segments involving a, the length of a side of the given square (each of these segments being itself constructible), a length $s/2$ was soon obtained. This will be included in the construction that now follows.

When several trials with different-sized squares turned out successfully, a proof was attempted and arrived at without much trouble. I shall give the construction first, and then the proof. Incidentally, have you noticed yet that all three of our cuts are of the same length?

Construction steps by which a given square may be cut into four parts with three cuts of equal length, and reassembled into an equilateral triangle:

In square $ABCD$, bisect AD and BC by E and N respectively. Call the side of the square a and the side of the proposed equilateral triangle s. As in *Figure* 6, draw a segment equal to a and by the familiar geometric construction determine $a/3$. As

Figure 6



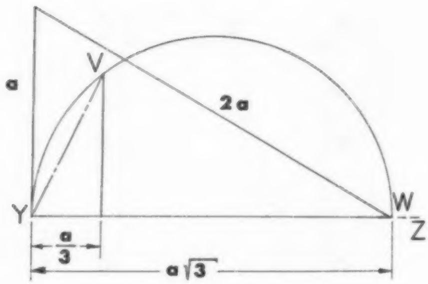


Figure 7

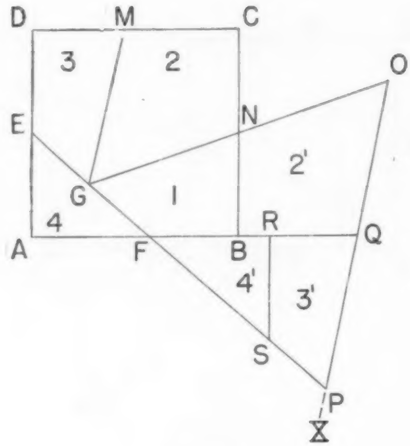


Figure 8

in *Figure* 7, draw segment a, and draw YZ perpendicular to it. Construct $2a$ from the end of a intersecting YZ. Then YW, by Pythagoras, will equal $a\sqrt{3}$. Mark off from Y a segment equal to $a/3$, and erect a perpendicular at this point. Construct a semicircle on YW, naming its intersection with this last perpendicular V. The segment YV will be equal to $\sqrt{a/3\cdot a\sqrt{3}}$, and this can be shown to be equal to $s/2$, by the following argument:

Equating the areas of the square and the triangle, we have:

$$a^2=\frac{s^2\sqrt{3}}{4},$$

$$a=\frac{s\sqrt[4]{3}}{2},$$

$$s=\frac{2a}{\sqrt[4]{3}},$$

$$\frac{s}{2}=\frac{a}{\sqrt[4]{3}},$$

$$\left(\frac{s}{2}\right)^2=\frac{a^2}{\sqrt{3}}=\frac{a}{3}\cdot a\sqrt{3},$$

$$\frac{s}{2}=\sqrt{a/3\cdot a\sqrt{.3}}$$

In the square locate point F as the intersection of the circle with center E and radius $s/2$ and line AB. With N as center and radius $s/2$ locate point G at the intersection of the circle with line EF. Now with G as center and radius $s/2$ let the circle intersect line DC and call the point of intersection M. If the cuts EF, GN, and GM are made, the square can be reassembled into an equilateral triangle.

The *proof* that follows is direct and easy to understand. I shall take the given square with the construction lines described in the last paragraph, add certain other lines forming a triangle, then prove that the triangle is equilateral and its parts are congruent to the corresponding parts of the square.

The proof follows (see *Figure* 8). Given: Square $ABCD$. *Given by construction:* Points E and N the midpoints of AD and BC respectively; Segments EF, GN, and GM all equal to $s/2$, this last length determined by our previous argument and *Figures* 6 and 7. Given also the following construction lines and segments which form a triangle composed of parts 1, 2′, 3′, and 4′: segment NO an extension of GN, and equal to it; OX parallel to GM; BQ an extension of AB and FP an extension of EF; $RQ=DM$, and RS perpendicular to RQ. Required to show that the triangle is equilateral.

Proof: (1) Quadrilateral 1 is common to both the square and the triangle.

(2) In quadrilaterals 2′ and 2 we have:

$\angle MGN = \angle NOQ$, $\angle NBQ = \angle NCM = 90°$; $\angle BNO = \angle GNC$; and side $BN =$ side NC. Hence the quadrilaterals are congruent, and $OQ = GN = s/2$.

(3) In quadrilaterals 3′ and 3 we have: $\angle RQP = \angle DMG$, $\angle QRS = \angle EDM = 90°$; $\angle DEG = \angle RSP$, and $RQ = DM$. Hence they are congruent, and $QP = GM = s/2$. Hence $GO = OP = s$.

(4) In triangles 4′ and 4 we have all the corresponding angles equal, and $RS = DE$ (from quadrilaterals 3′ and 3), $DE = AE$, hence $RS = AE$, and the triangles are congruent. Therefore $FS = EF = s/2$. Since $SP = EG$, we have

$$\begin{aligned} GP &= FS + (GF + SP) \\ &= s/2 + (GF + EG) \\ &= s/2 + s/2 \\ &= s. \end{aligned}$$

Thus $GP = GO = OP = s$, and the triangle is equilateral.

Notice that in the proof above no mention or use of angles of 60° was made, even though the original experimentation was partly based on these angles. This seems to me an example of deeper understanding through *doing*.

What about the converse, *i.e.*: Given an equilateral triangle, to cut it with three cuts into four parts which reassemble into a square? The reader may want to try this—it is not difficult. Further interesting relationships between segments *not yet used* easily come to light: BQ can be proved equal to AF; FQ is seen to have an important part in constructing the converse. Using the principles cited above from Problem IV, equating the areas of the equilateral triangle and the square again and this time solving for a in terms of s, and rearranging the result in suitable form for construction of a mean proportional are problems which you or your advanced geometry pupils may want to try.

"He who is unfamiliar with mathematics remains more or less a stranger to our time."—*E. Dillman.*

"Everything of importance has been said before by somebody who did not discover it."—*Alfred North Whitehead.*

The Weequahic configuration

E. R. RANUCCI, *American High School, San Salvador, El Salvador.*

An exercise in visualization in three-space.

BACK IN the good old days before "solid geometry" was criticized as much as it is today, we used to do a lot with orthographic projection in the solid geometry classes. We did not do the precise drawings that were turned out in the mechanical drawing classes—that was not our intention. The purpose of the unit was to exercise spatial imagery and to give visual intuitions a workout.

The customary approach to orthographic projection came first. Here a three dimensional object is imagined to be enclosed within a transparent rectangular solid. *Figure* 1 shows the views that are called the front view, top view, and end view. These two dimensional views are always placed as shown in *Figure* 1. For *Figure* 1 the following are true:

$$AB = CD = EF = GH$$

$$AC = BD = JK$$

$$EG = FH = IJ$$

Many such exercises were given. In some cases the orthographic projections were given, a sketch of the object being required. Sometimes the object was drawn and the orthographic projections were required. Due consideration was given the dotted line as contrasted to the solid line (*Fig.* 2).

After many such exercises students graduated to more sophisticated types. In these, two views were given, the third being required. Two of our favorites follow (*Fig.* 3).

The problem that developed into the "Weequahic" configuration (this all hap-

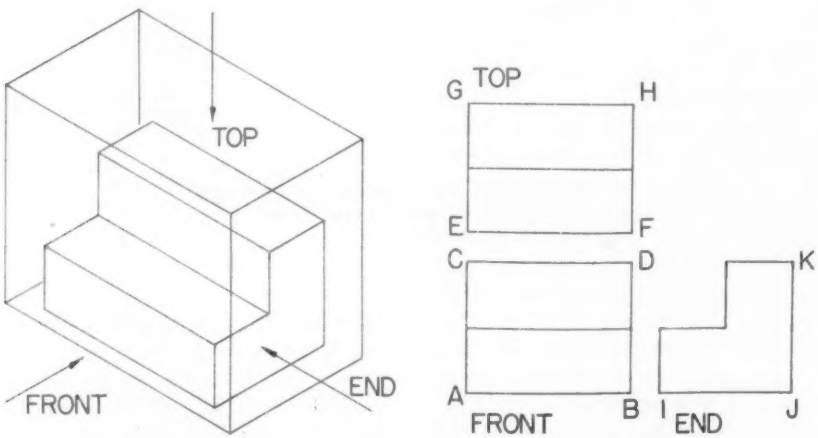


Figure 1

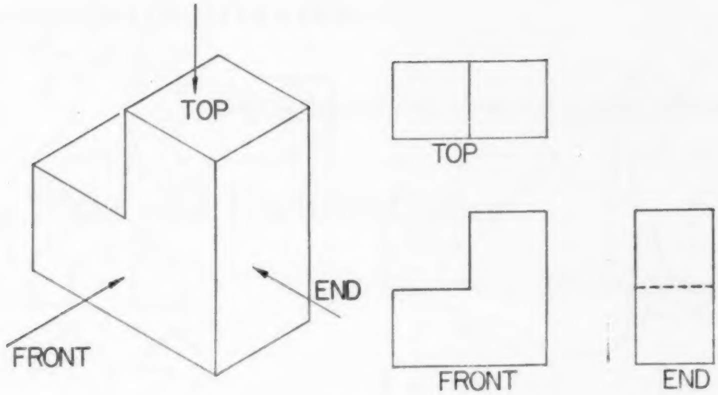


Figure 2

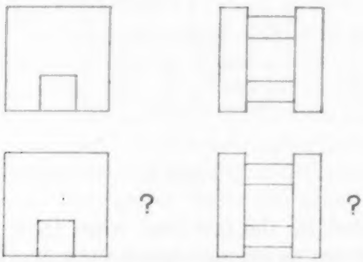


Figure 3

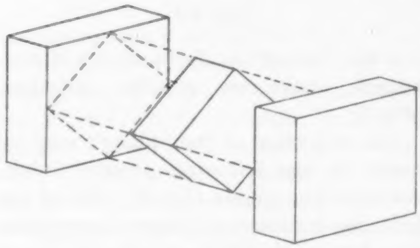


Figure 5

pened at Weequahic High School in Newark, New Jersey, between the years 1936 and 1956) started like this (*Fig.* 4):

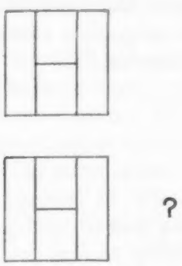


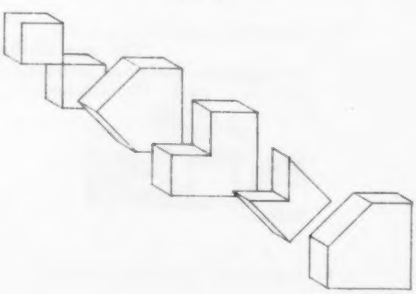
Figure 4

I had one object in mind (*Fig.* 5).

Little by little, however, students kept bringing in new solutions to the problem. At the last count, several *thousand* such configurations were shown to exist.

If the central figure in this "sandwich" is considered, here are variations on the possible form of the "filling" (*Fig.* 6).

Figure 6



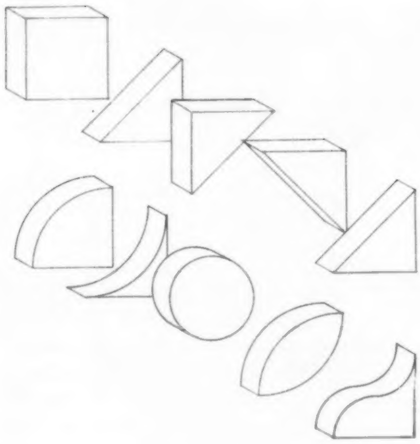


Figure 7

If the "bread" in the sandwich is considered, above are possible variations (*Fig.* 7).

The end view of the "filling" may be based on the following possible subdivisions of the square (*Fig.* 8). Any of the following outlines would show the requisite cross bars on the "H" needed in the center of the front and top views:

If only ten of the possible ends and ten of the central sections were placed in juxtaposition, 1000 such solids would be possible. Judicious interchange would increase this number considerably.

Some of the possible formations would be outlawed in cases where three adjoining faces were coplanar. This would be necessitated by the fact that when faces are coplanar no joining edges may be shown in orthographic projection.

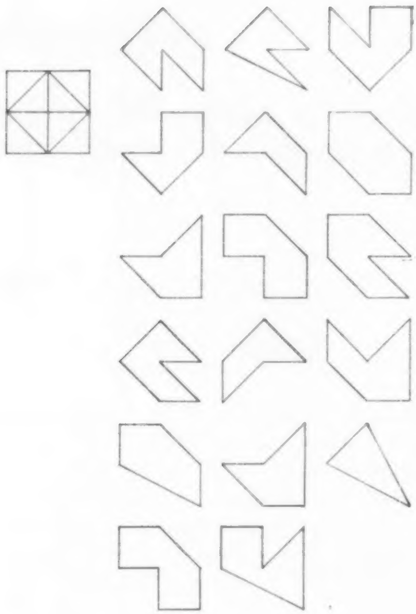


Figure 8

The moral of the story is: have more than one possible drawing in mind in exercises of this type. If you do not, the students will probably show *you* that they exist.

If you would like more information on orthographic projection consult a text in mechanical drawing.

"The true mathematician is always a good deal of an artist, an architect, yes, of a poet."—*A. Pringsheim.*

Edited by Howard Eves, University of Maine, Orono, Maine

Georg Mohr and Euclidis Curiosi

by Arthur E. Hallerberg, Illinois College, Jacksonville, Illinois

In 1673 a small geometrical treatise entitled *Compendium Euclidis Curiosi* (*Figure* 1 shows a reproduction of the title page of the work) was published in Dutch in Amsterdam. This work showed in effect that all constructions of Euclid's *Elements* are possible if one uses a straightedge and a compass restricted to one and the same opening throughout the entire construction.

No author's name was carried on this little booklet of twenty-four pages. When Joseph Moxon translated it from the Dutch into English and had it printed in London in 1677 (*Figure* 2 shows a reproduction of the title page of Moxon's translation), no additional information concerning the identity of the original author was included.

Although this work is a significant contribution in the development of the geometry of the fixed-compass [1],* both the original and the English translation appear to have received scant attention over the years. The purpose of this paper is to identify the author, to note why the booklet was written, and to consider some of the actual contents. The fact that the unnamed author received belated recognition for another geometrical contribution of even greater significance makes this of particular interest.

* Numbers in brackets refer to the notes and references at the end of the article.

Georg Mohr and "Euclides Danicus"

In 1928 the "rediscovery" of a small geometrical work entitled *Euclides Danicus* showed that Lorenzo Mascheroni had been preceded in the first presentation of the "geometry of the compass" by a little-known Danish mathematician, Georg Mohr [2]. In 1672 Mohr published in both Danish and Dutch a small booklet which contained a complete and elegant handling of the construction problems of Euclid using only the compass without the aid of

Figure 1

Compendium

EUCLIDIS CURIOSI:

Dat is,

Meetkonſtigh Paſſer-werck,

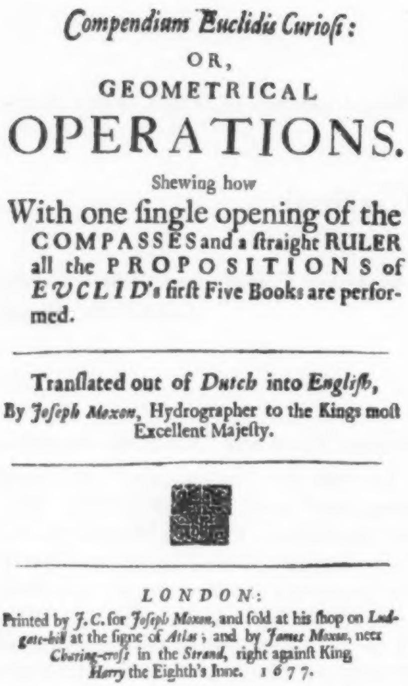
Hoe men

Met een gegeven opening van een Paſſer
en een Liniael, de Werck-ſtucken van Euclides,
ontbinden kan.

't Samen geſtelt door een Lief-hebber der ſelver Konſt.

T'AMSTERDAM,

By JOANNES JANSSONIUS van WAESBERGE, 1673.



Compendium Euclidis Curiosi:
OR,
GEOMETRICAL
OPERATIONS.
Shewing how
With one single opening of the COMPASSES and a straight RULER all the PROPOSITIONS of EUCLID's first Five Books are performed.

Translated out of *Dutch* into *English*,
By *Joseph Moxon*, Hydrographer to the Kings most Excellent Majesty.

LONDON:
Printed by *J. C.* for *Joseph Moxon*, and sold at his shop on *Ludgate-hill* at the signe of *Atlas*; and by *James Moxon*, neer *Charing-cross* in the *Strand*, right against King *Harry* the Eighth's Inne. 1677.

Figure 2

a straightedge. Since Mascheroni did not publish his *Geometria del Compasso* until 1797, the expression "Mascheroni construction" to indicate a construction carried out with the compass alone should really give place to "Mohr construction," although the former designation seems rather firmly entrenched in the literature.

The life of Georg Mohr

Some information on Mohr was given by the Danish mathematician Johannes Hjelmslev in the introduction to the reprint of the *Danicus* in 1928 (which included a German translation) and in an additional article three years later [3]. In brief, Hjelmslev gives the following details of Mohr's life.

Georg Mohr was born in Copenhagen on April 1, 1640. His father David was a merchant and an inspector of the hospital there. Early in life Mohr showed a special love for mathematics, and, recognizing that he would not be able to further his mathematical studies in Denmark, he left Copenhagen in 1662, at the age of twenty-two, and went to Holland.

Nothing is known of any formal schooling of Mohr in Holland. In 1672 he was in Amsterdam; in 1675 he was in England and Paris. In the spring of 1677 Mohr went from Paris to Holland and by 1681 was back in Copenhagen as an independent man of learning. Mohr married in 1687, and in 1691 he and his wife went to Zwolle, in Holland, where a son Peter Georg was born the next year. In the summer of 1695 Mohr and his family went to the estate of Walter Tschirnhaus in Kieslingswalde (Upper Lusatia), where he was to collaborate with Tschirnhaus in his studies.

Mohr died soon after, however, on January 26, 1697. He had traveled much during his life, and the many hardships endured in his travels may have contributed to his somewhat early death. During his earlier days in Holland he had been made a prisoner and was ill-treated by the French troops; at that time he lost all his goods and chattels including his manuscripts. Probably Mohr, like many others, was interested in Holland's war for independence.

In later life Mohr used the name Mohrendahl. His widow married Johann Friederich Schmied, a chemist in the service of Tschirnhaus. His son remained in Germany and Germanized the family name into Mohrenthal; he became a bookseller and publisher in Dresden.

Authorship of "Euclidis Curiosi"

Additional information on Mohr, seemingly unknown to Hjelmslev even in later years, comes from the correspondence made available when the *Gregory Tercentenary Volume* was published in 1939 [4]. This volume reprints several letters, in which Mohr's name is mentioned, from

John Collins to James Gregory. Of particular significance is the following, contained in a letter from Collins to Gregory written September 4, 1675. In this letter, after making reference to Mr. Walter Tschirnhaus, Collins wrote:

> . . . there being present with him a Dane named George Moorh who lately published in low Dutch, two little Bookes the one named Euclides Danicus where he pretends to perform all Euclids Problems with a paire of Compasses only without Ruler, and another intituled Euclides Curiosus, wherein with a Ruler and a forke (or the Compasses at one opening) he performs the same . . .

Apparently persons in recent years who have known of this letter have been unaware that copies of the *Curiosi* actually are still in existence, and those who have known of the *Curiosi* have not known of this letter.

For example, Hjelmslev pointed out at a later time that all of the copies of the Dutch edition of the *Danicus* which he had seen (he did not say how many this was) were bound together with a "small anonymous work," which we find through an intermediate reference refers to the *Curiosi* [5].

On the other hand, Joseph Hofmann referred to this letter in his work on Leibniz and added, "Unfortunately, until the present time we have not succeeded in bringing forth this second work of Mohr, just as Euclides Danicus was not brought to the light of day until 1928" [6].

On the basis of this letter (and also because of certain similarities in the handling of problems in the two works), the identification of Mohr as the author of the *Curiosi* seems fully justified.

Additional correspondence indicates that Mohr was also interested in other areas of mathematics besides geometry, particularly in the solution of the general cubic equation. It is impossible to identify or evaluate his contributions in the fields of analysis or algebra from the secondary sources available thus far. Hjelmslev quoted Mohr's son as stating that Mohr "had prepared three books on mathematics and philosophy, which met with the great approval of the scholars" [3]. In another letter from Collins to Gregory, October 19, 1675, Collins referred to Mohr's interest in the cubic equation and mentioned "a little treatise of that argument and of Fortifications etc. that he left in Holland to be printed in low Dutch" [4]. It is possible that this is the third work, along with the *Danicus* and the *Curiosi*.

The preface of the "Curiosi"

The *Curiosi* begins with a brief preface and then includes the description of thirty-two constructions. It is in the preface that the author indicates why he presents this work. Here the author's conjectures, explorations, and eventual triumph are so succinctly set forth that it seems appropriate to present them intact. The following is the English translation given by Moxon:

> It is very well known that the Judgements of men are changeable, and that at one time they will reject, through Ignorance, what at another time their Reason will cry up; and this many times proves prejudicial. Thus it hath happened to me: For having read in *Peter Ramus* his Dutch Geometry, printed at *Amsterdam* 1622. fol. 44. how that one *John Baptista* hath set forth a Book wherein with one single given opening of the Compasses, he answers all *Euclid's* Propositions and Operations. But that Book I could never yet get sight of. Therefore hath my curiosity prompted me to know the same; but when my thoughts fell upon the Two and twentieth Proposition of *Euclid's* first Book, how of three given right Lines, equal to three given right Lines, to make a Triangle, I confess I thought it was not to be done; and upon inquiry I found many skilful men of the same Opinion, also that the aforesaid Book never came forth, and that there were several *J. Baptista's* that have written &c. whereof *Marinus Bettinus* in his *Ærarium Philosophiae Mathematicae*, printed at *Bononia* 1648, makes mention: And if the áforesaid Book should have come forth, the aforesaid Author would doubtless have taken notice of it. But he hath performed several Propositions of *Euclid's* first Book, with once opening of the Compasses, as 1, 9, 10, 11, 12, and 31 Propositions. So also hath *Peter Wils*, in his *Geometrical Works*, printed at *Amsterdam* 1654, fol. 62. shown how in a Circle to finde all inscribed equal-sided Figures sides, from a figure of three Angles to one of twelve Angles: by keeping the Compasses at the same width the

Circle was made off, (which *Daniel Schwenter* also in his *Geometry* hath printed at *Nurenburg* 1641, fol. 208.) This Author does indeed shew the length of the Line: For example, he shews in a Pentagon, as in the Four and twentieth Proposition of this Treatise DE is done, but not on its place, as GD, &c. So hath also *Christopher Nottnagel* in his *Manuel Architecturae Militaris*, printed at *Wittenberg* 1659, fol. 177, shewn how a Regular Fortification, without Arithmetick and by one single opening of the Compasses, may be described on Paper; which is the Two and thirtieth Proposition of this Treatise.

These are all the Authors, ever came to my knowledge, that have written on this Subject; but they all use the Compasses opened to such a width, as one of the Lines (except *Bettinus* in some of his:) Neither do they scruple the using of a Line [7], though there be great difference in the Operation between a pair of Compasses and a Line, and a pair of Compasses alone; as may be seen in a Treatise named *Euclides Danicus*, set forth by *Georg Mohr*, and printed at Amsterdam 1672.

Upon these considerations have I found an impulse upon me to expose this Small Work to publick view, (especially since several have written on the Subject; and though all have aimed at it, yet none could hit the White, as aforesaid) How onely with one given width of the Compasses and a straight Ruler (which I will use at my own pleasure to prolong a Line with) to demonstrate the Propositions of *Euclid:* as you may see in this small Treatise.

And though these Propositions may be performed other ways, yet have I here set down only the most singular, that may be easiliest understood.

I would have added more Operations (as well how to perform the same with one given Line, and one given opening of the Compasses; as also with a given semi-Circle, or, &c.) as also how to describe Sundyals of what sort soever: but considering that all Flat or Plain Operations may be reduced from these, these shall suffice.

And if I finde this prove acceptable to you, then with good reason shall I publish something else of another nature.

A detailed consideration of the fixed-compass accomplishments of the various geometers referred to is not pertinent here (see reference [1]).

The rather casual reference to the *Danicus* is interesting, particularly if this is Mohr himself writing, as we believe it is. It can of course be noted that if the *Curiosi* had been better known over the years, the contents and true value of the *Danicus* might have been recognized much earlier.

Fixed-compass constructions in the "Curiosi"

We recall that in the preface the author states that "with one width of the compass and a straight ruler" he will "demonstrate the propositions of Euclid." Actually he presents twenty-nine basic constructions, three of which are not given in Euclid (8, 19b, and 29). The first of these is used quite often as an intermediate step in other constructions; the latter two are not referred to again.

Since there are forty-nine construction problems in Euclid, Mohr has not attempted every one of these. However, an analysis of the problems omitted reveals that all of these could be handled as in Euclid, by using the constructions Mohr has previously presented. Mohr makes no mention of this fact, however. He omits reference to problems of inscribing or circumscribing circles about polygons except for finding the center of the circle inscribed in a triangle. Mohr does not include proofs for the constructions, but they are all correct. Figures for the constructions are all given on a single plate at the beginning of the book.

Mohr's descriptions indicate that he is interested in performing the constructions quickly and simply—but he always makes provision for the case where the given opening might be less than one would arbitrarily choose if he had the opportunity. Of particular note is the fact that when some other circle is involved (as in drawing a tangent to a given circle from an outside point), he includes a case which assumes knowledge only of the center and the radius, without the circle being completely given. Of the various geometers who gave fixed-compass solutions for Euclid's problems, Mohr is the only one to do this.

Mohr closes his work with three additional propositions, the last two treating of Nottnagel's fortification problem. The other proposition is a geometrical problem concerning the dividing of a triangular

fish pond with a walk about it into two equal parts in a specially designated manner. No new operations are presented in its solution.

Since we hope that the idea of performing Proposition 22, Book I, of Euclid with a fixed-compass has sufficiently intrigued some of our readers to wonder at least on what main principle the construction depends, we will conclude by listing the first 14 propositions considered by Mohr and by including the constructions for several of the more difficult ones.

To distinguish the order of Mohr's propositions from the propositions of Euclid, we have coded these M-1, M-2, etc. The notation $A(r)$ means the circle drawn with center at A and with the fixed, arbitrary radius, r.

- M-1. To divide a given lines into two equal parts.
- M-2. To erect a perpendicular to a line from a given point in the given line.
- M-3. To construct an equilateral triangle on a given side.
- M-4. To erect a perpendicular to a line from a given point off the given line.
- M-5. Through a given point to draw a line parallel to a given line.
- M-6. To add two given line segments.
- M-7. To subtract a shorter segment from a given segment.
- M-8. Upon the end of a given line to place a given segment perpendicularly.
- M-9. To divide a line into any number of equal parts.
- M-10. Given two lines, to find the third proportional.
- M-11. Given three lines, to find the fourth proportional.
- M-12. To find the mean proportional to two given segments.
- M-13. To change a given rectangle into a square.
- M-14. To draw a triangle, given the three sides.

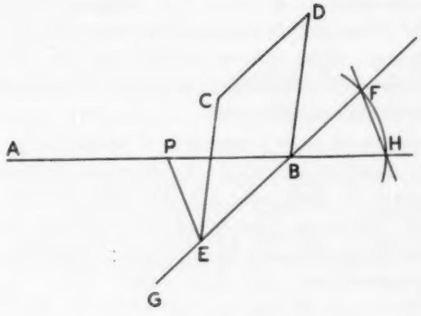
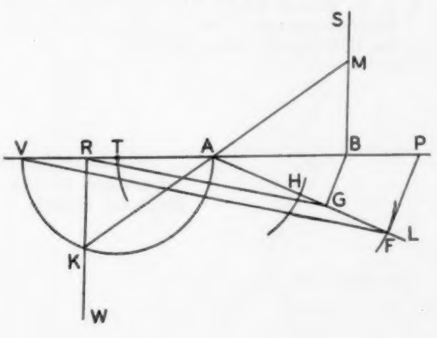


Figure 3

Solution of M-7: to subtract CD *from* AB (*Fig.* 3). Through B draw GB parallel to CD (by M-5). Connect DB and then draw CE parallel to DB (by M-5). $B(r)$ gives F and H. Through E draw the parallel to FH (by M-5), cutting AB at P. Then $AP = AB - CD$. (It is suggested to the reader that he attempt to adapt this construction to provide for M-6: to add two given segments.)

Solution of M-12: to find the mean proportional to two given segments, AB *and* BP (*Fig.* 4). BP can be placed in proper position by M-6. Make any angle PAL. $A(r)$ gives H and T. $H(r)$ gives F. $T(r)$ gives semicircle VA. Draw BG parallel to PF and GR parallel to FV (by M-5). Draw BS perpendicular to AB and RW perpendicular to AB (by M-2). RW cuts

Figure 4



semicircle in K. Join KA, cutting BS at M. Then BM is the required mean proportional. (The reader should analyze the construction to understand the purpose of transferring the ratio of the two given segments onto a segment of length $2r$.)

Solution of M-14: to draw a triangle given the three sides AB, AF, *and* FB (*Fig.* 5). Find m, the third proportional to $2AB$ and FB (by M-10); find n, the third proportional to $2AB$ and AF. Let $BG=(AB/2)+m-n$ (by M-6 and M-7). Next, find FG as the mean proportional to $(FB+GB)$ and $(FB-GB)$. Hence, at G erect the perpendicular and set off FG on this (by M-8), determining point F. Draw AF and BF to complete the triangle. (A student who is familiar with the Law of Cosines should be able to supply a proof for this construction—Mohr did not give one.)

Mohr's figure for the last construction is just what is given in *Figure* 5—the given sides and the completed triangle with the altitude GF. One is reminded of the comment of Jacob Steiner, made with reference to geometrical constructions in general, "... It is a very different matter actually to carry out the constructions, i.e., with the instruments in the hand, than it is to carry them through, if I may use the expression, simply by means of the tongue" [8]. Fortunately, for our purposes, it is sufficient to perform these constructions "by means of the tongue"!

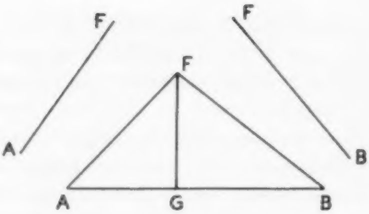


Figure 5

NOTES AND REFERENCES

1. A. E. HALLERBERG, "The Geometry of the Fixed-Compass," THE MATHEMATICS TEACHER, LII (April, 1959), 230–244.
2. For example, see N. A. COURT, "Mascheroni Constructions," THE MATHEMATICS TEACHER, LI (May, 1958), 370–372.
3. J. HJELMSLEV, "Beiträge zur Lebensbeschreibung von Georg Mohr," *Mathematisk-Fysiske Meddelelser (Danske Videnskabernes Selskab)*, Vol. 11, No. 4 (1931), 1–21.
4. JAMES GREGORY, *Tercentenary Memorial Volume*, edited by H. W. Turnbull (London, 1939), pp. 327 and 338.
5. J. HJELMSLEV, "Konstruktion ved Passer med fest Indstelling uden Brug af Lineal," *Matematisk Tidsskrift A* (1938), p. 77. Hjelmslev refers to an article by J. H. Schogt, "Om Georg Mohr's 'Euclides Danicus,'" same journal, pp. 34–36.
6. JOSEPH E. HOFMANN, *Die Entwicklungsgeschichte der Leibnizschen Mathematik* (München, 1949), p. 107. Several secondary sources appear to have misinterpreted the type of construction involved. O. Becker and J. Hofmann, *Geschichte der Mathematik* (Bonn, 1951), pp. 227–228 state, "the constructions by means of a fixed circle and ruler (*eines festen Kreises und des lineals*) appear in Mohr's Euclides curiosus (1672?)." L. Bieberbach, *Theorie der geometrischen Konstruktionen* (Basel, 1952), p. 150, says that the "Steiner constructions" were first considered in Mohr's *Curiosus*. It seems evident that all of these statements are based on Collins' letter and not on an actual examination of the *Curiosi*.
7. The word here and in the next phrase should be "ruler." Moxon had correctly translated the word *Liniael* on the title page.
8. J. STEINER, *Geometrical Constructions ...*, translated from the German by M. E. Stark (New York: Scripta Mathematica, 1950), pp. 64–65.

"Geometry has been, throughout, of supreme importance in the history of knowledge."—*Bertrand Russell.*

• NEW IDEAS FOR THE CLASSROOM

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Current proposals for new topics for the mathematics class suggest topics such as inequalities and polar equations. The articles below briefly describe ways of making these topics more meaningful. You can stimulate interest and help teachers introduce new topics if you will send your successful experiences to the editor.

Concerning simultaneous solutions of polar equations by the graphical method

by A. A. J. Hoffman and Roger Osborn, University of Texas, Austin, Texas

Some analytic geometry texts fail to emphasize the difference between the number of graphical intersections of two curves in polar coordinates and the number of simultaneous solutions of two equations in polar coordinates. For example, consider the equations $r=\frac{1}{2}$ and $r=\cos 2\theta$. When graphed on the same polar system they have eight "intersections" (see *Figure* 1).

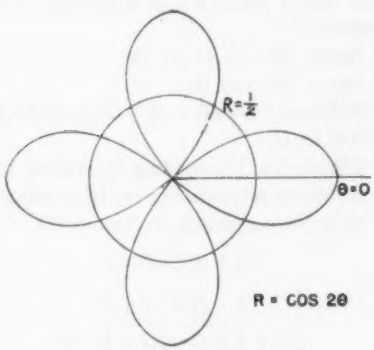


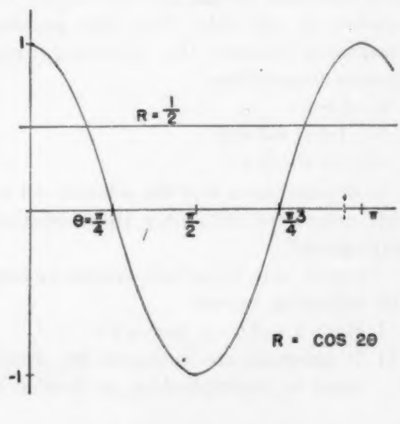
***Figure* 1**

However, solving the equations simultaneously yields only four simultaneous solutions:

$$\left(\frac{1}{2}, \frac{\pi}{6}\right), \quad \left(\frac{1}{2}, \frac{5\pi}{6}\right),$$
$$\left(\frac{1}{2}, \frac{7\pi}{6}\right), \quad \left(\frac{1}{2}, \frac{11\pi}{6}\right).$$

The reason for the difference can best be shown to the student by graphing the given equations in rectangular cartesian coordinates (see *Figure* 2).

Figure 2



It is easily seen that there are only four

"intersections" between 0 and 2π. It should be emphasized that the reason for the difference is as follows. In rectangular cartesian coordinates, each point has associated with it a unique ordered number pair (the coordinates of the point), and the equation of a locus which passes through a particular point is satisfied by the coordinates of that point. Conversely, any number pair which satisfies the equation must lie on the locus.

In polar coordinates, each point has associated with it an infinite number of ordered number pairs. That is, if a point is located by the coordinates (r, θ) then the same point has representations of the form $(r, \theta \pm n \cdot 2\pi)$ and $(-r, \theta + \pi \pm n \cdot 2\pi)$, where n is a positive integer. Not all representations of a point will satisfy the equation of a curve passing through that point. Therefore, in the example given, only four of the points have representations which satisfy both equations simultaneously.

An application of inequalities

by Don Wallin, Riverside-Brookfield High School, Riverside, Illinois

The axioms of inequalities have an interesting application in the solutions of verbal problems like this:

Al, Dave, Jim and Sam were comparing the number of used cars they each had sold that day:
a) Sam had sold more than Jim.
b) Between them, Al and Dave had sold just as many as Jim and Sam.
c) Al and Sam had not sold as many as Dave and Jim.
What is the order of their salesmanship ability in terms of the number of used cars sold?

To solve the problem let the initial of each salesman be the placeholder for the number of cars sold. Then the problem conditions become the following equations or inequalities:

a) $S > J$
b) $A+D=J+S$
c) $A+S<D+J$

Is it possible to find the solution set for four unknowns with only three relationships given?

To work with these inequalities we need the following axioms:

I If $a>b$ and $b>c$, then $a>c$.
II If unequals are increased by, diminished by, multiplied by, or divided by positive equals, the results are unequal in the same order.
III If unequals are subtracted from equals, the remainders are unequal in the reverse order.

There are various ways to combine the three relationships of the problem using the axioms of inequalities. With the operations below we arrive at a solution to the problem.

1. Apply Axiom II to the sum of statements (c) and (b).
2. Subtract $(S+D)=(S+D)$ according to Axiom II.
3. Divide by 2 according to Axiom II.
4. Subtract relationship (c) from relationship (b) according to Axiom III.

$$
\begin{array}{rcll}
A+S & < & D+J & (1) \\
A+D & = & J+S & \\
\hline
2A+S+D & < & 2J+S+D & \\
S+D & = & S+D & (2) \\
\hline
2A & < & 2J & \\
A & < & J \text{ or} & (3) \\
J & > & A &
\end{array}
$$

$$\begin{array}{rcl} A+D &=& J+S \\ A+S &<& D+J \\ \hline D-S &>& S-D \end{array} \qquad (4)$$

This inequality means that Dave's ability is greater than Sam's, if the left member of the inequality is positive and the right member negative. The absolute difference between their abilities is the same.

5. Add $(D+S)=(D+S)$ according to Axiom II
6. Divide by 2 according to Axiom II

$$\begin{array}{rcl} D-S &>& S-D \qquad (5) \\ D+S &=& D+S \\ \hline 2D &>& 2S \\ \hline D &>& S \qquad (6) \end{array}$$

7. Apply Axiom I to the relationships above, namely,

$D>S$, $S>J$, $J>A$, to obtain

$$D>S>J>A \qquad (7)$$

Or, in other words, Dave sold the greatest number of used cars, Sam next, Jim third, and Al sold the least number of cars.

Now try this problem:

Bob, Hank, Jack and Tom played tug-o-war one day and

a) although it was hard, Hank could just outpull Bob and Jack.

b) Hank and Bob together could just hold Tom and Jack, with neither side being more powerful.

c) If Jack and Bob change sides from their positions in statement (*b*), then Tom and Bob won rather easily.

Can you use the axioms of inequality to determine how these boys compare in strength?

A column of unofficial comment

Arithmetic and teacher preparation

by Albert E. Meder, Jr., Rutgers University, New Brunswick, New Jersey

In "Points and Viewpoints" in the November 1959 issue of THE MATHEMATICS TEACHER Professor Francis J. Mueller makes the important point that teachers of mathematics must have a knowledge of arithmetic, not mathematical background alone. I am sure all of my colleagues on the Commission on Mathematics would agree.

Professor Mueller also expresses the thought that the Report of the Commission "can very easily be construed to bolster" the view that "if the prospective teacher knew enough *arithmetic* to pass out of the eighth grade, then she knows enough *arithmetic* to go back and teach that subject at any grade level below the ninth." I am certain every one of my colleagues would deplore such a perversion of the Report and would applaud Professor Meuller for contending that it should not so be used.

But Professor Mueller's position seems to me to rest on selective quotation and arguments from silence. He asks: "Does the Report mean to imply that all or even most of the secrets of arithmetic understanding can be found in the study of algebra, geometry, and trigonometry?" and concludes, "Apparently so." But he overlooks the fact that the Commission regards this material as *background*—the word is used repeatedly—and that it says, too, "additional work in college mathematics would be highly desirable." I think my colleagues on the Commission would agree with me that it is almost axiomatic that additional work for students preparing for elementary teaching should include work in arithmetic. It is indeed unfortunate that we did not say so explicitly if the Report can be construed as Professor Mueller indicates.

Moreover, our recommendations for teaching an adequate *background* of mathematics to elementary teachers now in service indicate by implication, at least, that the members of the Commission did not hold, as Professor Mueller fears, that "all or even most of the secrets of arithmetic understanding" are to be found in this background. We said: "It is necessary that this subject matter be professionalized; in particular, that its relevance to the work of the elementary schools be stressed." Why would this have been added, had we held the view attributed to us?

It should be indicated, however, that algebra, geometry, and trigonometry taught in the spirit and manner suggested in the Commission Report would in fact almost certainly make a significant contribution to a prospective teacher's understanding of arithmetic; much more than could be expected were these subjects taught from a formal manipulative point of view.

But my main purpose in writing this note is to suggest that, since both adequate mathematical background and specific arithmetic knowledge are surely needed by elementary teachers, it would be well for

us not to quarrel among ourselves as to which is the more important, but rather to unite to secure enough time for mathematics in the teacher education curriculum so that both can be done.

In short, I don't think Professor Mueller is taking issue with the Commission Report. He is just using it as a point of departure to emphasize a different—but also vitally important—matter.

by Julius H. Hlavaty, Director, Mathematics Commission Program

May the writer associate himself wholly with Dean Meder's comments on Professor Mueller's viewpoint as expressed in the November 1959 issue of THE MATHEMATICS TEACHER?

Certainly it would be most unfortunate should the recommendations of the Commission on Mathematics be so misconstrued as Professor Mueller thinks they might be.

Anyone who has had the opportunity to survey—or even just to sense—the situation nationally with respect to the teaching of arithmetic will know the sad fact that there are many elementary school teachers who have had neither the three years of high school mathematics which the Commission recommends as a *minimum pretraining* experience in mathematics, nor the kind of professionalized course (or courses) in arithmetic which Professor Mueller recommends (and which is so well illustrated by his own book on the subject).

In contemplating this reality and to remedy this situation, we must develop ad hoc in-service courses which, of necessity, should stress Professor Mueller's approach. However, anyone who has tried such an approach (as the writer has) know how much more difficult it is when even a minimum amount of high school mathematics is lacking on the part of the participants.

How much brighter might the future be if the training of elementary school teachers, and especially their professionalized subject-matter courses in arithmetic, could be based on at least a three-year high school course of the kind recommended by the Commission!

Let us furthermore remember that the Commission's Program presupposes a sound training in preliminary mathematics: "... This includes skill in the operations at adult level ... and an understanding of the rationale of the computational processes. Understanding of a place system of writing numbers. . . . "—*Report of the Commission on Mathematics*, page 19.

It was not within the province of the assignment of the Commission to elaborate these suggestions. When the current interest in the improvement of mathematical instruction evolves sufficiently to make a marked impact on elementary education, not only will the Commission's Program be more easily realized, but a foundation for the training of future teachers will be that much more solid.

As Dean Meder says, let us work toward making it feasible for future teachers to get both background and specific training.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Finite Mathematical Structures, John G. Kemeny *et al.* (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1959). Cloth, xi+487 pp., $7.95.

This book is more exclusively a college-level book than its distinguished predecessor, *Finite Mathematics*, by John G. Kemeny *et al.*, which it overlaps somewhat but does not presuppose.

The book under review is likely to have several appeals to readers of THE MATHEMATICS TEACHER. 1) It presents in one volume a careful and reasonably elementary presentation of propositional logic, probability, vector spaces and matrices, linear programming, and Markov chains. 2) It employs the language of set theory and gives familiarity with supplementary devices such as tree diagrams. 3) It has a wealth of new applications which show the power and promise of these "modern" mathematical disciplines. Note that many of the applications lie in the physical sciences—a proof that the name "social science mathematics," which has been applied to some or all of these areas of mathematics, is too narrow a descriptive title.

We remarked that this book uses the language of set theory. Let us add that it does so with restraint. (For example, *function* is *not* defined as a kind of subset of a Cartesian product of sets—see p. 70.) This has some bearing on an important pedagogical problem. For the mathematics teacher is currently caught up in a conflict between a school of thought whom we may call the intensionalists, who consider *relation* and *property*, for example, as basic, primitive notions; and the extensionalists, who would reduce these and other concepts to set theory.

An illustration may clarify the nature of their disagreement. The extensionalists hold that the relation "greater than" (among the integers, say) is best defined by (is best construed as) the *set* of *all* ordered pairs, such as (3,1), (17, -4), etc.—all pairs in which the first component is the larger. In other words, they stress a set or extension of all possible cases. The intensionalists, on the other hand, argue that such a set of ordered pairs may mirror or clarify, but cannot displace the prior notion of "greater than." The disagreement thus transfers into the arena of set theory part of the old struggle between intuitionism and formalism. Pending a settlement of this conflict, it is probably fortunate that much of the literature, including the book under review, uses only the moderate species of set language and thereby avoids entanglement in this dispute.—*Carl H. Denbow, Ohio University, Athens, Ohio.*

On Mathematics and Mathematicians, Robert Edouard Moritz (New York: Dover Publications, Inc., 1959). Paper, v+410 pp., $1.95.

This delightful book is an unabridged and unaltered republication of the 1914 edition of *Memorabilia Mathematica.* It contains 1140 anecdotes, aphorisms, quotations from famous mathematicians, scientists and writers.

In its twenty-one chapters mathematicians will find encouragement, inspiration, and pleasure in reviewing the enormous achievements and dynamic personalities of great mathematicians. Teachers of mathematics and writers about mathematics will find this book a compendium of mathematical ideas and authentic quotations. The average layman, too, will find mathematics revealed as a dynamic and continually expanding science dedicated to the intellectual development and service of all mankind.

The annotated selections contained in this book were gathered from the works of three hundred authors, and deal with such topics as definitions and objects of mathematics; the teaching of mathematics; mathematics as a language or a fine art; modern mathematics; the relationship of mathematics to logic, philosophy and science; the nature of mathematics; the value of mathematics; research in mathematics. Special chapters are devoted to passages referring to the subject fields of arithmetic, algebra, geometry, the calculus and allied topics, and concepts of space and time.

The quotations, anecdotes, and aphorisms listed are, in the main, taken from the writings of great mathematicians of yesteryear. (Remember the book was first published in 1914.) However, many of the ideas presented by these mathematical giants are just as stimulating, penetrating, and applicable to today's science as they were when they were first presented.

A few quotations are:

"He who is unfamiliar with mathematics (literally, he who is a layman in mathematics) remains more or less a stranger to our time."—E. Dillman, *Die Mathematik die Fackelträgerin einer neuen Zeit* (Stuttgart, 1889), p. 39.

"One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics."—Novalis, *Schriften, Zweiter Teil* (Berlin, 1901), p. 223.

"All the modern higher mathematics is based on a calculus of operations, on laws of thought. All mathematics, from the first, was so in reality; but the evolvers of the modern higher calculus have known that it is so. Therefore elementary teachers who, at the present day,

persist in thinking about algebra and arithmetic as dealing with laws of number, and about geometry as dealing with laws of surface and solid content, are doing the best that in them lies to put their pupils on the wrong track for reaching in the future any true understanding of the higher algebras. Algebras deal not with laws of number, but with such laws of the human thinking machinery as have been discovered in the course of investigations on numbers. Plane geometry deals with such laws of thought as were discovered by men intent on finding out how to measure surface; and solid geometry with such additional laws of thought as were discovered when men began to extend geometry into three dimensions."—M. E. Boole, *Logic of Arithmetic* (Oxford, 1903), Preface, pp. 18–19.

"Every mathematical book that is worth reading must be read 'backwards and forwards,' if I may use the expression. I would modify Lagrange's advice a little and say, 'Go on, but often return to strengthen your faith.' When you come on a hard or dreary passage, pass it over; and come back to it after you have seen its importance or found the need for it further on." —George Chrystal, *Algebra, Part 2* (Edinburgh, 1889), Preface, p. 8.

For teachers, writers, laymen and embryonic mathematicians alike, this is a book well worth reading and retaining for future reference.—*Daniel W. Snader, Specialist for Mathematics, U. S. Office of Education.*

A Modern Introduction to College Mathematics, Israel Rose (New York: John Wiley and Sons, Inc., 1959). Cloth, xxi+529 pp., $6.50.

The point of view subscribed to by the author of this textbook is that ideas are basic in mathematics and that mathematics is best characterized, expressed, and understood in terms of its ideas. The ideas of mathematics which the author has reference to are: set, one-to-one correspondence, function, and relation. Emphasis is given to the sources (via the historical approach) and uses of these ideas and to the symbols and language in which they are expressed. These ideas are well used throughout the text to tie together the traditional subject matter of trigonometry, algebra, and analytic geometry. In addition these ideas are well used to introduce statistics and calculus.

An outstanding feature of the author's method of presentation is that in developing new concepts he consistently helps the reader to abstract the concepts instead of presenting them in completed form. The student is able to discover concepts for himself by this method of presentation.

This text is so written that it is possible to use it both with students desiring a "liberal arts" view of mathematics and with those preparing for a rigorous study of the subject. Questions and exercises are abundant throughout the text. Among the questions are those which are helpful for focusing and reviewing the concepts developed.

The book has a few weaknesses. First, the type used is small, fine, and light, and hence difficult to read. Second, the "boxing" of the formulas and definitions seems to place too great emphasis on the mere memorization of these ideas rather than on their development and understanding. Third, the author uses, at times, mnemonic devices in places where it seems contrary to his efforts at pointing out the logical and axiomatic nature of mathematics.—*Nancy C. Whitman, Wisconsin State College, Oshkosh, Wisconsin.*

The Modern Slide Rule, Stefan Rudolf (New York: The William-Frederick Press, 1959). Paper, 58 pp., $5.00.

This book is a complete text on simplified rules for fundamental slide-rule operations. It teaches problems common to all kinds of slide rules, considering the use of the principal *C*, *D*, *CI*, *A*, *B*, *K*, and *L* scales, certain special *CF*, *DF*, and *CIF* scales, trigonometric *S*, *T*, and *ST* scales, and Log Log, DF_m, and $\sqrt{\ }$ scales. In addition to a brief explanation of the operational technique, there are examples to explain the new and old ideas. Also, there is an excellent group of exercises, with answers, at the end of the book.

There are excellent drawings of several types of slide-rules at the beginning of the book, which illustrate the meaning of all the numbers for the various scales. These explain why the scales are divided in progressively diminishing parts.

The author gives a unique method of establishing the decimal point, although it isn't new. It does eliminate: (1) approximate calculating by "common sense"; and (2) rounding off factors to the smaller nearest result by mental arithmetic. Since it is necessary to have a basic knowledge of logarithms to understand the text as well as the operation and use of the slide rule, it would be much more convenient to use logarithmic characteristics or scientific notation than to show numbers that represent multiples of 10, 100, 1000, etc.

Under the rules for multiplication, it states:

"Set the hairline to a factor on the *D* scale.

"Set the left 1 (left index) on the *C* scale under the hairline.

"Move the hairline to the other factor on the *C* scale.

"Find the product under the hairline on the *D* scale."

In the second rule, it would be a little disconcerting to a beginning student trying to figure out one of the given examples, viz., $525 \times 0.38 = 199.5$, to find that this rule would place the other factor of the *C*-scale entirely off the *D* scale.

On the whole, the book is a well-written, easily understandable teaching text. The price of $5.00 is excessive, especially for high school students.—*William B. Lantz, Mansfield Senior High School, Mansfield, Ohio.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, the University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

When I teach geometry

by Hope H. Chipman, University School, Ann Arbor, Michigan

When I am teaching, especially in the junior high school and through the tenth grade, I try to give students the impression not only that mathematics is interesting, even exciting, but also that it is entirely possible to follow and understand it. I'll admit that in doing this I may not be giving a completely honest picture to some of the class members, but still, this is the viewpoint I try to put across.

In working toward this end, I usually introduce new material in class, so that the students and I develop the material together. I have studied the topic in detail myself and have a careful plan in mind. I often begin by saying, "You will need a pencil in hand and a piece of paper under it." If there is a figure, it is developed at the board and each student is encouraged to sketch his own. A typical question may be, "What do you think would be the next thing to prove? Jot your answer down on your paper."

In this way I help students to get away from the vagueness of response that may so easily result if there is a general question. If there are different responses I think a pupil is more likely to be encouraged to speak up for his own answer if he has been specific enough to write it down. "Do you think we now have enough material to show that two triangles are congruent? Answer Yes or No on your paper. If you said Yes, write down the pair of triangles and also the three pairs of congruent parts you are using."

With this type of activity I think I am able to give the students a major part in developing the new material, and to keep practically everyone alert and taking part. As we proceed in the course I give less help in the class development, perhaps only making a suggestion as to an auxiliary line to be used—leaving the rest of the explanation and proof to the class.

In my planning I have tried to remember what difficulties have arisen in other years so that I can either eliminate them in the explanation as it proceeds, or be ready to understand them quickly when they arise. For example, some pupils in every geometry class have trouble reading an angle with three letters. This seems a very easy thing to do, but experience has shown me that it is hard for some, so I always manage a variety of practice to help people acquire this skill.

Picking out corresponding sides and corresponding angles of congruent triangles is always difficult for some, too. Again, I develop a variety of practice along this line, being very careful to point out just which side is opposite each angle. With the new geometry material developed by the School Mathematics Study Group this difficulty should not arise, since congruent triangles and similar triangles are always lettered consistently

to point out the corresponding parts. That is, $\triangle ABC \cong \triangle DEF$ means that $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle B$, AB corresponds to DE, and so on.

Especially when students are beginning geometry, I want things to go smoothly. I'm eager to have them understand the material when it first appears. I want to avoid as much as possible that frustrated feeling, "I just don't get this," that ends in a mental block against the whole subject. In the assignment I suggest a careful review of the material covered in class, and I include a number of problems selected to use the new idea and drive it home.

Later in the year, of course, I throw students more on their own in studying new material. If I am teaching eleventh and twelfth grade students, I feel that it is part of my duty to give them practice in digging out textbook explanations on their own. I tell them often that reading and studying mathematics involves having a pencil and paper at hand for use in figuring out the intermediate steps which the authors have omitted.

As a rule I don't let a textbook restrict me unduly. If I think up a development different from that in the text and consider it an improvement, I use mine.

It is often necessary for the teacher to supply the simple one-step, one-question type of exercise needed to emphasize a point as discussion proceeds; to clear up the confusion shown by some facial expressions; to propose a numerical illustration. Such an exercise written into the margin of the book at the appropriate place may serve to recall next year an excellent illustration which may be forgotten in the interval.

Long lists of problems are needed to give the teacher an opportunity to choose the ones that seem especially helpful. Often I use only part of the list, giving easier ones to some class members and tough ones to other class members. No one text ever seems to have enough problem material. It is necessary to locate good problems in several texts and to develop problems that fit special situations which arise. I keep a list of these or a file for future use and use as many as are needed of the ones I like best to put across the idea, and then go along to new material. As far as I am concerned, problems are not necessarily kept for assignments. I use many of them in class. My idea is that problem sets should be used in any way or at any time that they serve to help in putting across the material.

Informal discussion among teachers about how we use problems is always interesting. I seem to be alone in my idea that it is neither profitable nor possible to have students write out several proofs every night. It is not possible in my case because I do not have time and patience to look them all over.

When we first begin to write out proofs of originals in my class, it is of course necessary to have everyone write out quite a few. Part of these I manage to check carefully myself. Part of them the class members check themselves; in this case everyone will have used identical lettering and a correct proof will be developed and written at the board, the whole class contributing. Alternate steps are pointed out and discussed and individual questions answered until I feel that everyone understands whether or not his proof is good and if not good, how it is wrong.

And soon I'm having one proof, or no proofs, written out and am making an assignment like this, "Study problems 3, 4, 5, 7, 11, and 12 to be ready to prove in class orally tomorrow. Bring your figures and enough notes so that when you get here you can quickly recall your plan and be able to give a full oral proof." I put it up to the students that I believe in this way we can cover more ground more quickly and can get more practice. I point out that by studying two or three exercises in the time it would take to write out one they are getting a wider variety of practice.

I honestly believe I get as far this way with my people with less frustration on

their part, and, to tell the truth, I just cannot face mountains of geometry proofs to correct night after night. To be sure there is the occasional boy or girl who shirks, but I can soon see who he is and get on his trail, and the first test shows him that the practice I advised is important.

Others are amazed by my statement of method—I am sure some think it an unwise plan—but for me it works. After all, geometry originals are fun to do, and it is much more fun to work at proving several than to spend time in writing out one or two.

I believe in using either two-column or paragraph proofs, as may seem most convenient. I do think that at first it will probably be easier for the students to do two-column proofs, since in that way they can more readily check the completeness of their proofs. You will do as you see fit about this, since a two-column proof is surely correct (if it is well done). This form of proof could be used until the end of the year, as probably many of us have been accustomed to doing. But I feel that it should be permissible to write a paragraph proof which, after all, is in line with the oral proof that is given in class. Also, in the paragraph form of proof it seems easier to give a more condensed, abbreviated proof. I approve of letting pupils do this as they become more adept, and as I become more familiar with their abilities and can tell which ones understand the material, even without a detailed proof. Of course I'll admit that all this fits in with my feeling that there can readily be too many detailed written proofs.

These points of my teaching method in geometry can be summarized as follows:

1. I try to lead young people to feel that mathematics is interesting and perfectly possible to understand.
2. I develop new material in class, particularly at the beginning of the year. Later in the year I give students more opportunity to study new material on their own. As new material is presented, I try to anticipate the learning difficulties that pupils are likely to have.
3. In classroom development, I have pupils keep paper and pencil ready to jot down answers to my questions, or to sketch a figure. This keeps them actively involved in the learning process.
4. I make a modest number of assignments for written proofs. Instead of many written proofs I ask pupils to study several problems to be ready to go over them orally in class.
5. Either a two-column or a paragraph proof is acceptable to me.

"The idea that aptitude for mathematics is rarer than aptitude for other subjects is merely an illusion which is caused by belated or neglected beginners."—*J. F. Herbart.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Thirty-eighth Annual Meeting

The Thirty-eighth Annual Convention of the National Council of Teachers of Mathematics will be held in the Hotel Statler Hilton, Buffalo, New York, on Thursday, Friday, and Saturday, April 21–23, 1960. Registration will begin Wednesday afternoon, April 20.

The opening general-session address on Thursday evening, "Mathematics and Human Knowledge," will be made by Carroll V. Newsom, president of New York University. The other general-session addresses, "Mathematics and Psychology" and "Human Intelligence and Mechanical Brains," will be given by R. Duncan Luce, University of Pennsylvania, and Tibor Rado, Ohio State University. The banquet address will be given by Dael Wolfle, executive officer, American Association for the Advancement of Science. President Harold P. Fawcett has selected as his retiring president's address "Guidelines in Mathematics Education," to be given at the luncheon.

At this meeting the Twenty-fifth Yearbook, *Instruction in Arithmetic*, under the editorship of Foster E. Grossnickle, will be presented. This Yearbook will provide the theme for several addresses during the program: "Arithmetic in Today's Culture," "Set, Scale, Symbol," "Definitions in Arithmetic," "Obtaining Greater Individual Differences," and "Guidance and Counseling in Arithmetic." At an early sectional meeting, editor Grossnickle will present "The Point of View of the Twenty-fifth Yearbook."

There are some forty-two sectional meetings planned to meet the interests and needs of all who may wish to attend the convention. Special attention will be given to a series of meetings for those interested in supervision. These meetings include in-service programs, book studies, a report of a research institute for elementary teachers, discussion of the role of city and state mathematics consultants, and a report on special services provided by the National Education Defense Act, by the U.S. Office of Education, and by the National Council of Teachers of Mathematics. Also, we will continue to have a series of lectures in mathematics and in the psychology of learning. Sessions of interest on teaching mathematics from the early grades of the elementary school through the undergraduate work of the college will be continued.

A demonstration with children in grades five and six is planned for those concerned with elementary school mathematics, to be taught by David Page, of the University of Illinois. Addresses will be "How to Improve the Elementary-School Mathematics Curriculum," "Thought Process of Sixth-Grade Pupils While Solving Verbal Problems in Arithmetic," "Depth in Arithmetic Learning—the Why and the How," "Teaching the Language of Per Cent," and "A New Look at Content and Its Placement in Elementary Mathematics."

In light of new developments in secondary-school mathematics, the following are some of the addresses to be given: "A High School Seminar in Modern Mathematics"; "A Report of SMSG Activities," to be followed by a panel of teachers dis-

cussing the use of the materials in the classroom; "Updating Mathematics Without a Revolution"; "The Madison Project, Algebra in Grades Five Through Eight"; "Certain Implications and Applications of the Set Concept"; "Vector Trigonometry"; "Probability and Statistics"; and "How Should we Evaluate the New Curriculum in Secondary Schools?" One sectional meeting is primarily concerned with general mathematics; those attending can expect to hear discussed, "Trends in Occupation Relating to Terminal Courses in High School Mathematics," and "Whither General Mathematics in the Junior High School?"

Of particular interest to those involved with student programs in the secondary schools are two meetings devoted to advanced placement. The latter of these will be a panel discussion by students who are or have been in advanced placement programs. Other kinds of programs to be discussed will include summer institutes, contests, clubs, and correspondence courses for gifted pupils. Helen Garstens, of the University of Maryland Mathematics Project, will teach a demonstration lesson using junior high school students.

The program has an international flavor too. There will be opportunity to hear about mathematics programs in Canada, Europe, the South Seas, Norway, Russia, and Australia.

One may wish to make some value judgments regarding approaches to teaching algebra and geometry. Addresses will be presented entitled, "A Comparison of Two Methods of Teaching Algebra in the Ninth Grade," "Content and Philosophy of the SMSG Elementary Algebra Curriculum," "The SMSG Geometry Program," and "In Defense of a Conservative Approach to Geometry."

At special request, one sectional meeting will be devoted to the mathematics education of elementary teachers. However, junior high school and senior high school teacher education has not been neglected. As usual, we have some very excellent sectional meetings planned by our committees in research, evaluation, the gifted student, television, and international relations.

Three laboratory sessions are available, one for elementary school teachers, one for junior high school teachers, and one for senior high school teachers. However, these will emphasize ideas, rather than extensive construction of materials.

For those who register early, there will be special tours to observe electronic mathematical operations at the Remington Rand Division of the Sperry Rand Company and at the International Business Machines Company. Many undoubtedly will want to visit Niagara Falls during their stay.

May you plan now to be among the some 2500 elementary, secondary, and college mathematics teachers who are expected to attend this convention!

"The social sciences mathematically developed are to be the controlling factors in civilization."—*W. F. White*, A Scrap-book of Elementary Mathematics, *Chicago, 1908, p. 208.*

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangle, Board of Education, Salt Lake City 11, Utah

NINETEENTH CHRISTMAS MEETING

December 27–30, 1960
Arizona State College, Tempe, Arizona
William D. Ray, Arizona State College, Tempe, Arizona

Other professional dates

Mathematics Section, New York Society for Experimental Study of Education

March 12, 1960
Grace Dodge Hall, Teachers College, Columbia University, New York, New York
John A. Schumaker, Secretary, Montclair State College, Upper Montclair, New Jersey

Ohio Wesleyan Conference on Modern Trends in Mathematics

March 12, 1960
Ohio Wesleyan University, Delaware, Ohio
Lyman C. Peck, Ohio Wesleyan University, Delaware, Ohio

Chicago Elementary Teachers' Mathematics Club

March 14, 1960
Toffenetti's Restaurant, private dining room, 65 West Monroe Street, Chicago, Illinois
Ramona H. Goldblatt, Burley School, Chicago, Illinois

Georgia Mathematics Council

March 18, 1960
City Auditorium, Atlanta, Georgia
James P. Brown, Southwest High School, 3116 Sewell Road, S.W., Atlanta 11, Georgia

Mathematics Section, Maryland State Teachers Association

March 26, 1960
Towson State Teachers College, Towson 4, Maryland
W. Edwin Freeny, President, 507 Milford Mill Road, Pikesville 8, Maryland

The Association of Mathematics Teachers of New Jersey

March 26, 1960
Trenton State College, Trenton, New Jersey
Gail B. Koplin, 14 Schuyler Drive, Clark, New Jersey

Association of Teachers of Mathematics of New York City

March 26, 1960
Room 846, Washington Irving High School, 40 Irving Place, New York, New York
Lee Batch, New Utrecht High School, 17th Avenue and 80th Street, Brooklyn, New York

Illinois Council of Teachers of Mathematics

March 26, 1960, East St. Louis, Illinois
April 2, 1960, Normal, Illinois
April 9, 1960, Macomb, Illinois
April 15, 1960, Charleston, Illinois
April 16, 1960, Carbondale, Illinois
April 30, 1960, Cicero, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

The Ontario Association of Teachers of Mathematics and Physics

April 19–20, 1960
Ontario College of Education, Toronto, Ontario
Father John C. Egsgard, St. Michael's College School, Toronto 10, Ontario

The Nebraska Section of the National Council of Teachers of Mathematics

April 30, 1960
Lincoln Public Schools Administration Building, 720 South Twenty-Second Street, Lincoln, Nebraska
Monte S. Norton, 720 South Twenty-Second Street, Lincoln, Nebraska

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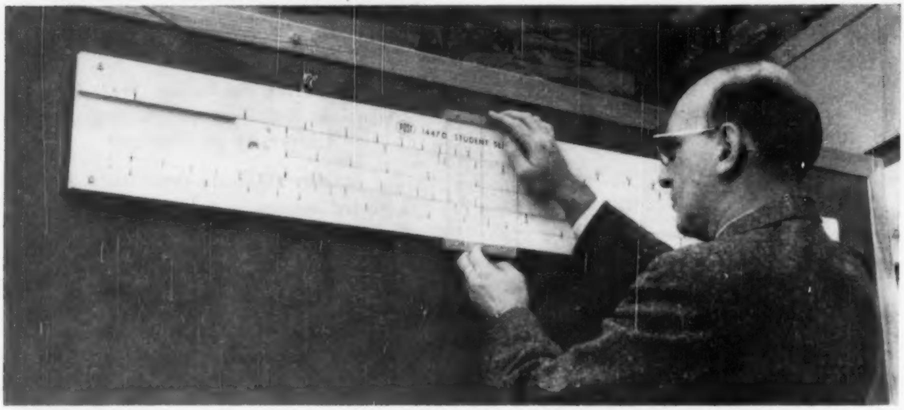
The facts about "Bargain" slide rules

by **ROBERT JONES,** *Manager of Educational Sales, Frederick Post Company*



Teaching suggestion: Many math and engineering instructors find this scaled-up model of the POST Student Slide Rule an efficient teaching aid

Unless your students know what to look for in a slide rule, they will be tempted by "bargain" sales messages to buy one of the "less expensive rules" retailing for 75c to $1.25. Here are the short-comings that are certain to show up in these so-called "specials".

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- **Materials.** Because they're made of low cost materials, bargain slide rules often warp as a result of climate changes, the readings become undependable. Slides either stick or become loose. Cursors do not operate properly.

A "bargain" like this is no bargain at all.

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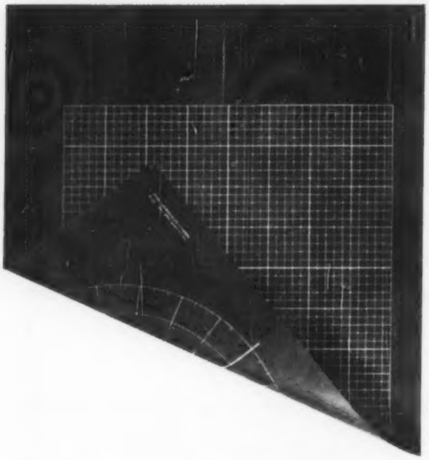

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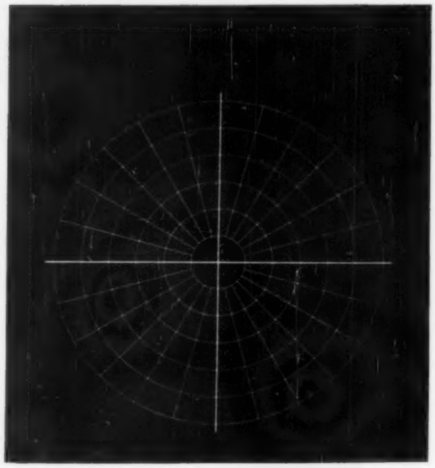

EB199 Polar Graph Chart ....
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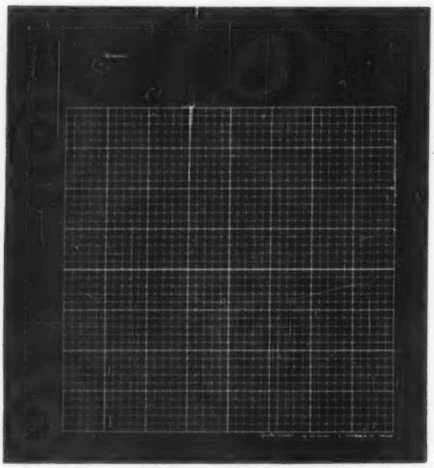

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